

# ON THE SOLUTION OF STOCHASTIC OPTIMIZATION AND VARIATIONAL PROBLEMS IN IMPERFECT INFORMATION REGIMES

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**Abstract.** We consider the solution of a stochastic convex optimization problem  $\mathbb{E}[f(x; \theta^*, \xi)]$  over a closed and convex set  $X$  in a regime where  $\theta^*$  is unavailable and  $\xi$  is a suitably defined random variable. Instead,  $\theta^*$  may be obtained through the solution of a learning problem that requires minimizing a metric  $\mathbb{E}[g(\theta; \eta)]$  in  $\theta$  over a closed and convex set  $\Theta$ . Traditional approaches have been either sequential or direct variational approaches. In the case of the former, this entails the following steps: (i) a solution to the learning problem, namely  $\theta^*$ , is obtained; and (ii) a solution is obtained to the associated computational problem which is parametrized by  $\theta^*$ . Such avenues prove difficult to adopt particularly since the learning process has to be terminated finitely and consequently, in large-scale instances, sequential approaches may often be corrupted by error. On the other hand, a variational approach requires that the problem may be recast as a possibly non-monotone stochastic variational inequality problem in the  $(x, \theta)$  space; but there are no known first-order stochastic approximation schemes currently available for the solution of this problem. To resolve the absence of convergent efficient schemes, we present a coupled stochastic approximation scheme which simultaneously solves *both* the computational and the learning problems. The obtained schemes are shown to be equipped with almost sure convergence properties in regimes when the function  $f$  is either strongly convex as well as merely convex. Importantly, the scheme displays the optimal rate for strongly convex problems while in merely convex regimes, through an averaging approach, we quantify the degradation associated with learning by noting that the error in function value after  $K$  steps is  $\mathcal{O}(\sqrt{\ln(K)/K})$ , rather than  $\mathcal{O}(\sqrt{1/K})$  when  $\theta^*$  is available. Notably, when the averaging window is modified suitably, it can be seen that the original rate of  $\mathcal{O}(\sqrt{1/K})$  is recovered. Additionally, we consider an online counterpart of the misspecified optimization problem and provide a non-asymptotic bound on the average regret with respect to an offline counterpart. In the second part of the paper, we extend these statements to a class of stochastic variational inequality problems, an object that unifies stochastic convex optimization problems and a range of stochastic equilibrium problems. Analogous almost-sure convergence statements are provided in strongly monotone and merely monotone regimes, the latter facilitated by using an iterative Tikhonov regularization. In the merely monotone regime, under a weak-sharpness requirement, we quantify the degradation associated with learning and show that expected error associated with  $\text{dist}(x_k, X^*)$  is  $\mathcal{O}(\sqrt{\ln(K)/K})$ . Preliminary numerics demonstrate the performance of the prescribed schemes.

**Key words.** stochastic optimization, stochastic variational inequality, stochastic approximation, learning

**AMS subject classifications.**

**1. Introduction.** In the last two decades, robust optimization [7, 8] approaches have grown in relevance when decision-makers are faced with optimization problems with uncertain parameters. Succinctly, in such an approach, given an uncertainty set that captures the realizations assumed by such a parameter, the *robust* solution represents the *worst-case* over this set of realizations. Naturally, an appropriate choice of such an uncertainty set is crucial and as the availability of data reaches levels hitherto unseen, there is growing interest in data-driven approaches [9] for constructing such sets. Our interest is in closely related yet distinct settings driven by data in which the point estimate of a parameter may be obtained through a learning problem, suitably defined through the aggregation of data. We provide two instances of such problems:

(i) *Portfolio optimization.* Portfolio optimization problems prescribe the optimal constructions of portfolios over a set of assets, for which the mean and covariance of returns are not necessarily known. Traditional approaches have assumed that such returns are available while more recent robust optimization models have

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utilized factor-based models in constructing uncertainty sets [22, 11, 10]. An alternate, and possibly less conservative, data-driven model of such a problem that employs a point estimate of the mean and covariance matrix requires the solution of two coupled problems: (1) A portfolio optimization problem parametrized by  $(\theta^*, \Sigma^*)$  representing the mean and covariance matrix of returns; and (2) A learning problem that utilizes data to obtain the best  $(\theta^*, \Sigma^*)$ .

(ii) *Power systems operation.* The operation of power grids relies on the solution of hourly (or more frequent) commitment and dispatch problems, each of which is reliant on a range of parameters that are often uncertain. These parameters include supply-side information regarding capacity of wind-power as well as load forecasts. Recently robust optimization approaches have proved to be exceedingly popular [26, 49, 43]. An alternate formulation is given by the following two coupled problems: (1) An economic dispatch problem parametrized by  $\theta^*$ , a vector that captures the unknown supply and demand side parameters; and (2) A learning problem that computes  $\theta^*$  through the accumulation of data.

We believe that such coupled formulations have broad applicability beyond merely the settings mentioned above in (i) and (ii). They may also find application in inventory control problems with stochastic demand [36, 1, 44, 2], robust network design [33], robust routing in communication networks [23], amongst others. To recap the difference between the two problem frameworks, it can be seen that (R-Opt), a robust optimization framework, minimizes the worst-case of the optimal value  $f(x; \theta)$  over the uncertainty set  $\mathcal{U}_\theta$  while (L-Opt) considers the joint solution of an optimization problem in  $x$ , parametrized by  $\theta^*$ , where  $\theta^*$  is a solution to a learning problem with a metric  $g(\theta)$ . The following formulations may provide a clearer comparison:

<div style="border: 1px solid black; padding: 10px;"> <p>R-Opt      minimize    <math>\max_{\theta \in \mathcal{U}_\theta} f(x; \theta)</math>  subject to    <math>x \in X</math>.</p> </div>	<div style="border: 1px solid black; padding: 10px;"> <p>L-Opt      minimize    <math>f(x; \theta^*)</math>  <math>\quad \quad \quad x \in X</math>  minimize    <math>g(\theta)</math>  <math>\quad \quad \quad \theta \in \Theta</math></p> </div>
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We consider regimes where the function  $f(x; \theta)$  is a convex expected-value function and the resulting problem is given by the following:

$$(\mathcal{P}_x^o(\theta^*)) \quad \min_{x \in X} \mathbb{E}[f(x, \xi(\omega); \theta^*)],$$

where  $X \subseteq \mathbb{R}^n$  is a closed and convex set,  $\xi : \Omega \rightarrow \mathbb{R}^d$  is a  $d$ -dimensional random variable defined on a probability space  $(\Omega, \mathcal{F}_x, \mathbb{P}_x)$ ,  $f : X \times \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}$  is a real-valued function, and  $\theta^*$  denotes an  $m$ -dimensional vector of parameters. Estimating such parameters often requires the resolution of a suitably defined learning problem, given by a stochastic optimization problem  $(\mathcal{L}_\theta)$ , and defined next:

$$(\mathcal{L}_\theta) \quad \min_{\theta \in \Theta} g(\theta) \triangleq \mathbb{E}[g(\theta; \eta)],$$

where  $\Theta \subseteq \mathbb{R}^m$  is a closed and convex set,  $\eta : \Lambda \rightarrow \mathbb{R}^p$  is a random variable defined on a probability space  $(\Lambda, \mathcal{F}_\theta, \mathbb{P}_\theta)$ , and  $g : \Theta \times \Lambda \rightarrow \mathbb{R}$  is a real-valued function. When one considers the joint problem of learning and optimization, then there are at least two obvious approaches that immediately emerge as possibilities:

**(a) Sequential approach:** Consider an inherently serial process wherein the first stage incorporates a model/parameter specification phase based on statistical learning while the second stage leverages these findings in developing and solving the actual optimization problem of interest. Such an ordering relies on the learning problems being relatively small and tractable compared to the optimization problems, ensuring that accurate solutions are available within a reasonable time period. Strictly speaking, if one terminates the learning process prematurely with an estimator  $\hat{\theta}$ , the resulting estimator is essentially corrupted by

error in that  $\hat{\theta} \neq \theta^*$ . This error propagates into the solution  $\hat{x}$  of the computational problem, denoted by  $\mathcal{P}_x^o(\hat{\theta})$  and the associated gap might be quite significant. Note that unless the learning problem is solvable via a finite termination algorithm, such a approach cannot provide asymptotic statements but can, at best, provide approximate solutions. Consequently, an inherently serial process reliant on a prematurely truncated learning scheme often fails to provide accurate solutions to the computational problem.

**(b) Variational approach:** Under suitable convexity and differentiability requirements, the following holds:

$$x^* \text{ solves } (\mathcal{P}_x^o(\theta^*)) \text{ and } \theta^* \text{ solves } (\mathcal{L}_\theta),$$

if and only if  $(x^*, \theta^*)$  is a solution to the (stochastic) variational inequality problem  $\text{VI}(Z, F)$  [15] where

$$Z \triangleq X \times \Theta \text{ and } H(z) \triangleq \begin{pmatrix} \mathbb{E}[\nabla_x f(x, \theta; \xi)] \\ \mathbb{E}[\nabla_\theta g(\theta; \eta)] \end{pmatrix}.$$

Recall that  $z^*$  is a solution to  $\text{VI}(Z, F)$  if  $(z - z^*)^T F(z) \geq 0$  for all  $z \in Z$ . Furthermore, if  $x^*$  and  $\theta^*$  denote solutions to  $(\mathcal{P}_x^o(\theta^*))$  and  $(\mathcal{L}_\theta)$ , respectively, then an oft-used avenue in obtaining a solution  $(x^*, \theta^*)$  entails obtaining a solution to  $\text{VI}(Z, F)$ . However, unless rather strong assumptions are imposed, the map  $H$  is not necessarily monotone, precluding the use of recently developed stochastic approximation schemes for solving monotone stochastic variational inequality problems [25, 32, 47], extragradient-based variants [27, 48], and accelerated approaches [13].

**Simultaneous approach:** This paper is motivated by the inadequacy of available approaches and, more generally, the absence of *asymptotically convergent schemes with provable non-asymptotic rates*. We present a framework where the learning and the computational problems are solved **simultaneously** via a joint set of stochastic approximation schemes. Such an avenue has several advantages. First, under such an approach, one can provide rigorous statements of asymptotic convergence of the obtained estimators for both, the solution to the computational problem and the associated learning problem. Second, error bounds on the expected error can be provided for a fixed number of steps under a regime with constant and diminishing steplengths. Third, the statements may be extended to the variational regime in which the computational problem is given by the variational counterpart of  $(\mathcal{P}_x^o(\theta^*))$ , given by  $(\mathcal{P}_x^v(\theta^*))$ ; such a problem requires an  $x^* \in X$  such that

$$(\mathcal{P}_x^v(\theta^*)) \quad \mathbb{E}[F(x^*, \xi(\omega); \theta^*)]^T (x - x^*) \geq 0, \quad \forall x \in X,$$

where  $X \subseteq \mathbb{R}^n$  is a closed and convex set,  $\xi : \Omega \rightarrow \mathbb{R}^d$  is a  $d$ -dimensional random variable defined on a probability space  $(\Omega, \mathcal{F}_x, \mathbb{P}_x)$ ,  $F : X \times \mathbb{R}^d \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a real-valued continuous mapping. Note that when  $F(x^*, \xi(\omega); \theta^*) \triangleq \nabla_x f(x^*; \xi; \theta^*)$ , this reduces to a convex optimization problem. Furthermore, the choice of using a variational problem, rather than merely an optimization problem, is founded on the need to model a variety of multiagent settings complicated by a breadth of strategic interactions, ranging from purely cooperative to distinctly noncooperative [16].

**1.1. Related decision-making models.** While unaware of the availability of general purpose tools that can resolve precisely such problems, we describe settings where such questions have assumed relevance: **Adaptive control [5]:** In tracking problems in adaptive control [3], the authors consider a perturbation approach for analyzing a adaptive tracking algorithm and consider three estimation schemes, specifically least mean squares (LMS) scheme, its recursive variant (RLMS), and the Kalman filter (which requires some distributional assumptions on the noise). First, much of this treatment is in the unconstrained regime

with tractable (often quadratic estimation objectives), allowing for deriving closed-form (and often linear) update rules. Second, when the noise in the estimation process is Gaussian, the Kalman filter provides a minimum variance estimator. If on the other hand, the noise is non-Gaussian, then the Kalman filter provides the optimal linear estimator (in the sense that no linear filter provides smaller variance). In fact, these assumptions often form the basis of most adaptive control algorithms (cf. [37] and [34] for a discussion adaptive control and stochastic approximation.) Our focus is on static stochastic problems with far less assumptions on the nature of the problem and the associated distributions. Specifically, we allow for more general stochastic convex objectives (or monotone maps in the context of VIs) in either the optimization or the learning problem, allow for convex feasibility sets for both the optimization or the learning problems, and impose relatively mild moment assumptions on the noise (unlike the Gaussian assumptions that are necessary in some of the estimation models).

**Iterative learning control:** A related avenue lies in iterative learning control (ILC) has its roots in the studies by Uchiyama [46] and Arimoto et al. [4]. ILC [39] is a form of tracking control employed for repetitive control problems, instances being chemical batch processes, robot arm manipulators, and reliability testing rigs. Our problem is more restrictive in its focus (static problems) but allow for more general settings in terms of nonlinearity and the underlying distributional requirements.

**Multi-armed bandit problems:** The multi-armed bandit (MAB) problem considers the question of how to play given a collection of slot machines faced by a gambler. Each machine provides a random reward from a distribution specific to that machine. The gambler aims to maximize the expected sum of rewards earned through a sequence of lever pulls. The total discounted reward is maximized by the index policy that pulls the bandit having greatest value of the Gittins index [21]. In effect, the reward function needs to be learnt while optimizing the system. There has been significant research on such problems over the last several decades, including on the question of computation [30] and finite-time analysis [6].

Finally, related questions also been studied in revenue management where [14] examined the devastating effect of learning with an incorrect model while maximizing revenue.

**1.2. Outline and contributions.** Broadly speaking, this paper focuses on the development of *stochastic approximation schemes* that generate iterates  $\{x_k\}$  and  $\{\theta_k\}$  and makes the following contributions. (i) In Section 2, we prove the a.s. convergence of the produced iterates to the prescribed solutions and derive error bounds in a standard and an averaging regime. In particular, we quantify the degradation in the convergence rate from introducing an additional learning phase; (ii) Section 2 concludes with a precise non-asymptotic bound on the average regret associated with employing the proposed scheme instead of an offline algorithm; (iii) In Section 3, we extend the a.s. convergence results to accommodate stochastic variational inequality problems, rather than merely convex optimization problems. Error analysis is carried out under a suitably defined growth property; (iv) In Section 4, we provide some supporting numerics and conclude in Section 5. Finally, throughout the paper, we use  $\|x\|$  to denote the Euclidean norm of a vector  $x$ , i.e.,  $\|x\| = \sqrt{x^T x}$  and  $\Pi_K$  to denote the Euclidean projection operator onto a set  $K$ , i.e.,  $\Pi_K(x) \triangleq \operatorname{argmin}_{y \in K} \|x - y\|$ .

**2. Stochastic optimization problems with imperfect information.** In this section, we focus on examining  $(\mathcal{P}_x^o(\theta^*))$  under various assumptions. We begin by stating the coupled stochastic approximation scheme and providing the necessary assumptions in Section 2.1. Convergence analysis of the presented scheme is provided in Section 2.2 while diminishing and constant steplength rate analysis is performed in Section 2.3. We conclude with a discussion of an online algorithm with the associated bounds on the decay of average regret in Section 2.4.

**2.1. Algorithm statement and assumptions.** As mentioned in the previous section, we propose a set of coupled stochastic approximation schemes for computing  $x^*$  and  $\theta^*$ .

ALGORITHM 1 (Coupled SA schemes for stochastic optimization problems). **Step 0.** Given  $x_0 \in X, \theta_0 \in \Theta$  and sequences  $\{\gamma_{k,x}, \gamma_{k,\theta}\}$ ,  $k := 0$

**Step 1.**

$$\begin{aligned} (\text{Opt}_k) \quad & x^{k+1} := \Pi_X \left( x^k - \gamma_{k,x} (\nabla_x f(x^k; \theta^k) + w^k) \right), \quad k \geq 0 \\ (\text{Learn}_k) \quad & \theta^{k+1} := \Pi_\Theta \left( \theta^k - \gamma_{k,\theta} (\nabla_\theta g(\theta^k) + v^k) \right), \quad k \geq 0 \end{aligned}$$

where  $w^k \triangleq \nabla_x f(x^k; \theta^k, \xi^k) - \nabla_x f(x^k; \theta^k)$  and  $v^k \triangleq \nabla_\theta g(\theta^k; \eta^k) - \nabla_\theta g(\theta^k)$ .

**Step 2.** If  $k > K$ , stop; else  $k := k + 1$ , go to Step. 1.

We begin by stating an assumption on the functions  $f$  and  $g$ .

ASSUMPTION 1 (Problem properties, A1-1). Suppose the following hold:

- (i) For every  $\theta \in \Theta$ ,  $f(x; \theta)$  is strongly convex and continuously differentiable with Lipschitz continuous gradients in  $x$  with convexity constant  $\mu_x$  and Lipschitz constant  $L_x$ , respectively.
- (ii) For every  $x \in X$ , the gradient  $\nabla_x f(x; \theta)$  is Lipschitz continuous in  $\theta$  with constant  $L_\theta$ .
- (iii) The function  $g(\theta)$  is strongly convex and continuously differentiable with Lipschitz continuous gradients in  $\theta$  with convexity constant  $\mu_\theta$  and Lipschitz constant  $C_\theta$ , respectively.

Under Assumption (A1-1), the coupled problem admits a unique solution, as shown next.

LEMMA 2.1 (Solvability). Consider the problems  $(\mathcal{P}_x^\circ(\theta^*))$  and  $(\mathcal{L}_\theta)$  and suppose assumption (A1) holds. Then  $(\mathcal{P}_x^\circ(\theta^*))$  and  $(\mathcal{L}_\theta)$  collectively admit a unique solution.

*Proof.* This follows from the strong convexity of  $g$  over  $\Theta$  and the strong convexity of  $f(\bullet; \theta)$  over  $X$ .  $\square$

Additionally, we make the following assumptions on the steplength sequences employed in the algorithm.

ASSUMPTION 2 (Steplength requirements, A2-1). Let  $\{\gamma_{k,x}\}$  and  $\{\gamma_{k,\theta}\}$  be chosen such that:

- (i)  $\sum_{k=0}^{\infty} \gamma_{k,x} = \infty$ ,  $\sum_{k=0}^{\infty} \gamma_{k,x}^2 < \infty$
- (ii)  $\gamma_{k,\theta} = \gamma_{k,x} L_\theta^2 / (\mu_x \mu_\theta)$ .

We define a new probability space  $(Z, \mathcal{F}, \mathbb{P})$ , where  $Z \triangleq \Omega \times \Lambda$ ,  $\mathcal{F} \triangleq \mathcal{F}_x \times \mathcal{F}_\theta$  and  $\mathbb{P} \triangleq \mathbb{P}_x \times \mathbb{P}_\theta$ . We use  $\mathcal{F}_k$  to denote the sigma-field generated by the initial points  $(x^0, \theta^0)$  and errors  $(w^l, v^l)$  for  $l = 0, 1, \dots, k-1$ , i.e.,  $\mathcal{F}_0 = \{(x^0, \theta^0)\}$  and  $\mathcal{F}_k = \{(x^0, \theta^0), ((w^l, v^l), l = 0, 1, \dots, k-1)\}$  for  $k \geq 1$ . We make the following assumptions on the filtration and errors.

ASSUMPTION 3 (A3). Let the following hold:

- (i)  $\mathbb{E}[w^k | \mathcal{F}_k] = 0$  and  $\mathbb{E}[v^k | \mathcal{F}_k] = 0$  a.s. for all  $k$ .
- (ii)  $\mathbb{E}[\|w^k\|^2 | \mathcal{F}_k] \leq \nu_x^2$  and  $\mathbb{E}[\|v^k\|^2 | \mathcal{F}_k] \leq \nu_\theta^2$  a.s. for all  $k$ .

We conclude this subsection by stating three results (without proof) that will be subsequently employed in developing our convergence statements. The first two of these are relatively well-known super-martingale convergence results (cf. [41, Lemma 10, Pg. 49–50])

LEMMA 2.2. Let  $v_k$  be a sequence of nonnegative random variables adapted to  $\sigma$ -algebra  $\mathcal{F}_k$  and such that

$$\mathbb{E}[v_{k+1} | \mathcal{F}_k] \leq (1 - u_k) v_k + \beta_k \quad \text{for all } k \geq 0 \quad \text{almost surely,}$$

where  $0 \leq u_k \leq 1$ ,  $\beta_k \geq 0$ , and  $\sum_{k=0}^{\infty} u_k = \infty$ ,  $\sum_{k=0}^{\infty} \beta_k < \infty$  and  $\lim_{k \rightarrow \infty} \frac{\beta_k}{u_k} = 0$ . Then,  $v_k \rightarrow 0$  a.s.

LEMMA 2.3. Let  $v_k$ ,  $u_k$ ,  $\beta_k$  and  $\gamma_k$  be non-negative random variables adapted to  $\sigma$ -algebra  $\mathcal{F}_k$ . If  $\sum_{k=0}^{\infty} u_k < \infty$ ,  $\sum_{k=0}^{\infty} \beta_k < \infty$  and

$$\mathbb{E}[v_{k+1} | \mathcal{F}_k] \leq (1 + u_k)v_k - \gamma_k + \beta_k \quad \text{for all } k \geq 0 \quad \text{almost surely.}$$

Then,  $\{v_k\}$  is convergent and  $\sum_{k=0}^{\infty} \gamma_k < \infty$  almost surely. Finally, we present a contraction result reliant on monotonicity and Lipschitz continuity requirements (cf. [17, Theorem 12.1.2, Pg. 1109]).

LEMMA 2.4. Let  $H : K \rightarrow \mathbb{R}^n$  be a mapping that is strongly monotone over  $K$  with constant  $\mu$ , and Lipschitz continuous over  $K$  with constant  $L$ . If  $q \triangleq \sqrt{1 - 2\mu\gamma + \gamma^2 L^2}$ , then for any  $\gamma > 0$ , we have that for any  $x, y$ , we have  $\|\Pi_K(x - \gamma H(x)) - \Pi_K(y - \gamma H(y))\| \leq q\|x - y\|$ .

**2.2. Almost-sure convergence.** Our first convergence result shows that under the prescribed assumptions, Algorithm 1 generates a sequence of iterates that converges to the unique solution.

PROPOSITION 1 (**Almost-sure convergence under strong convexity of  $f$** ). Suppose (A1-1), (A2-1) and (A3) hold. Let  $\{x^k, \theta^k\}$  be computed via Algorithm 1. Then,  $x^k \rightarrow x^*$  and  $\theta^k \rightarrow \theta^*$  a.s. as  $k \rightarrow \infty$ , where  $\theta^*$  denotes the unique solution of  $(\mathcal{L}_\theta)$  and  $x^*$  denotes the unique solution to  $(\mathcal{P}_x^o(\theta^*))$ .

*Proof.* Note that  $x^* = \Pi_X(x^* - \gamma_{k,x} \nabla_x f(x^*; \theta^*))$ . Then, by the nonexpansivity of the Euclidean projector,  $\|x^{k+1} - x^*\|^2$  may be bounded as follows:

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &= \|\Pi_X(x^k - \gamma_{k,x}(\nabla_x f(x^k; \theta^k) + w^k)) - \Pi_X(x^* - \gamma_{k,x} \nabla_x f(x^*; \theta^*))\|^2 \\ &\leq \|(x^k - x^*) - \gamma_{k,x}(\nabla_x f(x^k; \theta^k) - \nabla_x f(x^*; \theta^*)) - \gamma_{k,x} w^k\|^2. \end{aligned}$$

By adding and subtracting  $\gamma_{k,x} \nabla_x f(x^*, \theta^k)$ , this expression can be further expanded as follows:

$$\begin{aligned} &\|(x^k - x^*) - \gamma_{k,x}(\nabla_x f(x^k; \theta^k) - \nabla_x f(x^*; \theta^k)) - \gamma_{k,x}(\nabla_x f(x^*; \theta^k) - \nabla_x f(x^*; \theta^*)) - \gamma_{k,x} w^k\|^2 \\ &= \|(x^k - x^*) - \gamma_{k,x}(\nabla_x f(x^k; \theta^k) - \nabla_x f(x^*; \theta^k))\|^2 + \gamma_{k,x}^2 \|\nabla_x f(x^*; \theta^k) - \nabla_x f(x^*; \theta^*)\|^2 + \gamma_{k,x}^2 \|w^k\|^2 \\ &\quad - 2\gamma_{k,x}[(x^k - x^*) - \gamma_{k,x}(\nabla_x f(x^k; \theta^k) - \nabla_x f(x^*; \theta^k))]^T (\nabla_x f(x^*; \theta^k) - \nabla_x f(x^*; \theta^*)) \\ &\quad - 2\gamma_{k,x}[(x^k - x^*) - \gamma_{k,x}(\nabla_x f(x^k; \theta^k) - \nabla_x f(x^*; \theta^k))]^T w^k + 2\gamma_{k,x}^2 (\nabla_x f(x^*; \theta^k) - \nabla_x f(x^*; \theta^*))^T w^k. \end{aligned}$$

By leveraging the fact that  $\mathbb{E}[w^k | \mathcal{F}_k] = 0$ , we have

$$(2.1) \quad \mathbb{E}[\|x^{k+1} - x^*\|^2 | \mathcal{F}_k] \leq \text{Term 1} + \text{Term 2} + \text{Term 3} + \gamma_{k,x}^2 \mathbb{E}[\|w^k\|^2 | \mathcal{F}_k],$$

where **Terms 1 – 3** are defined as follows:

$$\begin{aligned} \text{Term 1} &\triangleq \|(x^k - x^*) - \gamma_{k,x}(\nabla_x f(x^k; \theta^k) - \nabla_x f(x^*; \theta^k))\|^2, \\ \text{Term 2} &\triangleq \gamma_{k,x}^2 \|\nabla_x f(x^*; \theta^k) - \nabla_x f(x^*; \theta^*)\|^2, \\ \text{and Term 3} &\triangleq -2\gamma_{k,x}[(x^k - x^*) - \gamma_{k,x}(\nabla_x f(x^k; \theta^k) - \nabla_x f(x^*; \theta^k))]^T (\nabla_x f(x^*; \theta^k) - \nabla_x f(x^*; \theta^*)). \end{aligned}$$

By Lemma 2.4 and (A1-1), it follows that

$$(2.2) \quad \text{Term 1} \leq (1 - 2\gamma_{k,x}\mu_x + \gamma_{k,x}^2 L_x^2) \|x^k - x^*\|^2.$$

Furthermore, the Lipschitz continuity of  $\nabla_x f(x^*; \theta)$  in  $\theta$  (A1-1) allows for deriving the following bound:

$$(2.3) \quad \mathbf{Term\ 2} \leq \gamma_{k,x}^2 L_\theta^2 \|\theta^k - \theta^*\|^2.$$

Finally, **Term 3** can be bounded by invoking the Cauchy-Schwarz inequality, Lemma 2.4, (A1-1) and the triangle inequality, we obtain

$$(2.4) \quad \begin{aligned} & 2\gamma_{k,x} \|(x^k - x^*) - \gamma_{k,x}(\nabla_x f(x^k; \theta^k) - \nabla_x f(x^*; \theta^k))\| \|\nabla_x f(x^*; \theta^k) - \nabla_x f(x^*; \theta^*)\| \\ & \leq 2\gamma_{k,x} \sqrt{1 - 2\gamma_{k,x}\mu_x + \gamma_{k,x}^2 L_x^2} \|x^k - x^*\| L_\theta \|\theta^k - \theta^*\| \\ & \leq 2\gamma_{k,x} L_\theta \|x^k - x^*\| \|\theta^k - \theta^*\| \\ & \leq \gamma_{k,x} \mu_x \|x^k - x^*\|^2 + \gamma_{k,x} (L_\theta^2 / \mu_x) \|\theta^k - \theta^*\|^2, \end{aligned}$$

where the last inequality follows from  $2a^T b \leq \|a\|^2 + \|b\|^2$ . Combining (2.1), (2.2), (2.3) and (2.4), we get

$$(2.5) \quad \begin{aligned} \mathbb{E}[\|x^{k+1} - x^*\|^2 \mid \mathcal{F}_k] & \leq (1 - \gamma_{k,x}\mu_x + \gamma_{k,x}^2 L_x^2) \|x^k - x^*\|^2 \\ & \quad + (\gamma_{k,x} L_\theta^2 / \mu_x + \gamma_{k,x}^2 L_\theta^2) \|\theta^k - \theta^*\|^2 + \gamma_{k,x}^2 \nu_x^2. \end{aligned}$$

Recall that  $\theta^*$  satisfies the fixed point relationship  $\theta^* = \Pi_\Theta(\theta^* - \gamma_{\theta,k} \nabla_\theta g(\theta^*))$ , which, together with non-expansivity of the Euclidean projector, allows for deriving the following bound on  $\|\theta^{k+1} - \theta^*\|^2$ :

$$\begin{aligned} \|\theta^{k+1} - \theta^*\|^2 & = \|\Pi_\Theta(\theta^k - \gamma_{\theta,k}(\nabla_\theta g(\theta^k) + v^k)) - \Pi_\Theta(\theta^* - \gamma_{\theta,k} \nabla_\theta g(\theta^*))\|^2 \\ & \leq \|\theta^k - \theta^* - \gamma_{\theta,k}(\nabla_\theta g(\theta^k) - \nabla_\theta g(\theta^*)) - \gamma_{\theta,k} v^k\|^2 \\ & = \|\theta^k - \theta^* - \gamma_{\theta,k}(\nabla_\theta g(\theta^k) - \nabla_\theta g(\theta^*))\|^2 + \gamma_{\theta,k}^2 \|v^k\|^2 - 2(\theta^k - \theta^* - \gamma_{\theta,k}(\nabla_\theta g(\theta^k) - \nabla_\theta g(\theta^*)))^T v^k. \end{aligned}$$

By taking conditional expectations and by recalling that  $\mathbb{E}[v^k \mid \mathcal{F}_k] = 0$ , we obtain the following bound:

$$(2.6) \quad \begin{aligned} \mathbb{E}[\|\theta^{k+1} - \theta^*\|^2 \mid \mathcal{F}_k] & \leq \|\theta^k - \theta^* - \gamma_{k,\theta}(\nabla_\theta g(\theta^k) - \nabla_\theta g(\theta^*))\|^2 + \gamma_{k,\theta}^2 \mathbb{E}[\|v^k\|^2 \mid \mathcal{F}_k] \\ & \leq q_{k,\theta}^2 \|\theta^k - \theta^*\|^2 + \gamma_{k,\theta}^2 \nu_\theta^2, \end{aligned}$$

where  $q_{k,\theta} \triangleq \sqrt{1 - 2\gamma_{k,\theta}\mu_\theta + \gamma_{k,\theta}^2 C_\theta^2}$ . Next, by adding (2.5) and (2.6) and by invoking (A2-1), we obtain the following bound.

$$\begin{aligned} & \mathbb{E}[\|x^{k+1} - x^*\|^2 \mid \mathcal{F}_k] + \mathbb{E}[\|\theta^{k+1} - \theta^*\|^2 \mid \mathcal{F}_k] \\ & \leq (1 - \gamma_{k,x}\mu_x + \gamma_{k,x}^2 L_x^2) \|x^k - x^*\|^2 + (q_{k,\theta}^2 + \gamma_{k,x} L_\theta^2 / \mu_x + \gamma_{k,x}^2 L_\theta^2) \|\theta^k - \theta^*\|^2 + \gamma_{k,x}^2 \nu_x^2 + \gamma_{k,\theta}^2 \nu_\theta^2 \\ & = (1 - \gamma_{k,x}\mu_x + \gamma_{k,x}^2 L_x^2) \|x^k - x^*\|^2 + (1 - \gamma_{k,x} L_\theta^2 / \mu_x + \gamma_{k,x}^2 (L_\theta^2 + L_\theta^4 C_\theta^2 / (\mu_x^2 \mu_\theta^2))) \|\theta^k - \theta^*\|^2 \\ & \quad + \gamma_{k,x}^2 \nu_x^2 + \gamma_{k,x}^2 \nu_\theta^2 L_\theta^4 / (\mu_x^2 \mu_\theta^2) \\ & \leq (1 - \alpha\gamma_{k,x} + \beta\gamma_{k,x}^2) (\|x^k - x^*\|^2 + \|\theta^k - \theta^*\|^2) + \delta\gamma_{k,x}^2, \end{aligned}$$

where  $\alpha = \min\{\mu_x, L_\theta^2 / \mu_x\}$ ,  $\beta = \max\{L_x^2, L_\theta^2 + L_\theta^4 C_\theta^2 / (\mu_x^2 \mu_\theta^2)\}$  and  $\delta = \nu_x^2 + \nu_\theta^2 L_\theta^4 / (\mu_x^2 \mu_\theta^2)$ . From (A2-1), we have that  $\sum_{k=0}^\infty (\alpha\gamma_{k,x} - \beta\gamma_{k,x}^2) = \infty$ ,  $\sum_{k=0}^\infty \delta\gamma_{k,x}^2 < \infty$ , and

$$\lim_{k \rightarrow \infty} \frac{\delta\gamma_{k,x}^2}{\alpha\gamma_{k,x} - \beta\gamma_{k,x}^2} = 0.$$

Then, by invoking the super-martingale convergence theorem (Lemma 2.2), we have that  $\|x^k - x^*\|^2 + \|\theta^k - \theta^*\|^2 \rightarrow 0$  a.s. as  $k \rightarrow \infty$ , which implies that  $x^k \rightarrow x^*$  and  $\theta^k \rightarrow \theta^*$  a.s. as  $k \rightarrow \infty$ .  $\square$

Next we weaken the strong convexity requirement on the function  $f$  through the following assumption.

**ASSUMPTION 4 (A1-2).** *Suppose the following holds in addition to (A1-1 (ii)) and (A1-1 (iii)).*

- (i) *For every  $\theta \in \Theta$ ,  $f(x; \theta)$  is convex and continuously differentiable with Lipschitz continuous gradients in  $x$  with Lipschitz constant  $L_x$ .*

Furthermore, we make the following assumptions on the steplength sequences employed in the algorithm.

**ASSUMPTION 5 (A2-2).** *Let  $\{\gamma_{k,x}\}$ ,  $\{\gamma_{k,\theta}\}$  and some constant  $\tau \in (0, 1)$  be chosen such that:*

- (i)  $\sum_{k=0}^{\infty} \gamma_{k,x}^{2-\tau} < \infty$  and  $\sum_{k=0}^{\infty} \gamma_{k,\theta}^2 < \infty$ ,  
 (iii)  $\sum_{k=0}^{\infty} \gamma_{k,x} = \infty$  and  $\sum_{k=0}^{\infty} \gamma_{k,\theta} = \infty$ ,  
 (iii)  $\beta_k = \frac{\gamma_{k,x}^\tau}{2\gamma_{k,\theta}\mu_\theta} \downarrow 0$  as  $k \rightarrow \infty$ .

Proceeding as in the previous result, we present a convergence result under these weakened conditions.

**THEOREM 1 (Almost-sure convergence under convexity of  $f$ ).** *Suppose (A1-2), (A2-2) and (A3) hold. Suppose  $X$  is bounded and the solution set  $X^*$  of  $(\mathcal{P}_x^o(\theta^*))$  is nonempty. Let  $\{x^k, \theta^k\}$  be computed via Algorithm 1. Then,  $\theta^k \rightarrow \theta^*$  a.s. as  $k \rightarrow \infty$ , and  $x^k$  converges to a random point in  $X^*$  a.s. as  $k \rightarrow \infty$ , where  $\theta^*$  denotes the unique solution of  $(\mathcal{L}_\theta)$  and  $X^*$  denotes the solution set of  $(\mathcal{P}_x^o(\theta^*))$ .*

*Proof.* By the nonexpansivity of the Euclidean projector, we have for any  $x^* \in X^*$  that

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &= \|\Pi_X(x^k - \gamma_{k,x}(\nabla_x f(x^k; \theta^k) + w^k)) - \Pi_X(x^*)\|^2 \\ &\leq \|(x^k - x^*) - \gamma_{k,x} \nabla_x f(x^k; \theta^k) - \gamma_{k,x} w^k\|^2. \end{aligned}$$

By adding and subtracting  $\gamma_{k,x} \nabla_x f(x^*, \theta^k)$ , this expression can be further expanded as follows:

$$\begin{aligned} &\|(x^k - x^*) - \gamma_{k,x} \nabla_x f(x^k; \theta^*) - \gamma_{k,x}(\nabla_x f(x^k; \theta^k) - \nabla_x f(x^k; \theta^*)) - \gamma_{k,x} w^k\|^2 \\ &= \|(x^k - x^*) - \gamma_{k,x} \nabla_x f(x^k; \theta^*)\|^2 + \gamma_{k,x}^2 \|\nabla_x f(x^k; \theta^k) - \nabla_x f(x^k; \theta^*)\|^2 + \gamma_{k,x}^2 \|w^k\|^2 \\ &\quad - 2\gamma_{k,x} [(x^k - x^*) - \gamma_{k,x} \nabla_x f(x^k; \theta^*)]^T (\nabla_x f(x^k; \theta^k) - \nabla_x f(x^k; \theta^*)) \\ &\quad - 2\gamma_{k,x} [(x^k - x^*) - \gamma_{k,x} \nabla_x f(x^k; \theta^*)]^T w^k + 2\gamma_{k,x}^2 (\nabla_x f(x^k; \theta^k) - \nabla_x f(x^k; \theta^*))^T w^k. \end{aligned}$$

Noting that  $\mathbb{E}[w^k | \mathcal{F}_k] = 0$ , we have

$$(2.7) \quad \mathbb{E}[\|x^{k+1} - x^*\|^2 | \mathcal{F}_k] \leq \mathbf{Term\ 1} + \mathbf{Term\ 2} + \mathbf{Term\ 3} + \gamma_{k,x}^2 \mathbb{E}[\|w^k\|^2 | \mathcal{F}_k],$$

where **Terms 1 – 3** are defined as follows:

$$\begin{aligned} \mathbf{Term\ 1} &\triangleq \|(x^k - x^*) - \gamma_{k,x} \nabla_x f(x^k; \theta^*)\|^2, \\ \mathbf{Term\ 2} &\triangleq \gamma_{k,x}^2 \|\nabla_x f(x^k; \theta^k) - \nabla_x f(x^k; \theta^*)\|^2, \\ \text{and } \mathbf{Term\ 3} &\triangleq -2\gamma_{k,x} [(x^k - x^*) - \gamma_{k,x} \nabla_x f(x^k; \theta^*)]^T (\nabla_x f(x^k; \theta^k) - \nabla_x f(x^k; \theta^*)). \end{aligned}$$

By invoking the convexity of  $f(x; \theta)$  in  $x$  and the gradient inequality (see A1-2), we have that

$$\begin{aligned} \mathbf{Term\ 1} &= \|x^k - x^*\|^2 + \gamma_{k,x}^2 \|\nabla_x f(x^k; \theta^*)\|^2 - 2\gamma_{k,x} (x^k - x^*)^T \nabla_x f(x^k; \theta^*) \\ &\leq \|x^k - x^*\|^2 + \gamma_{k,x}^2 \|\nabla_x f(x^k; \theta^*)\|^2 - 2\gamma_{k,x} (f(x^k; \theta^*) - f(x^*; \theta^*)) \\ &\leq \|x^k - x^*\|^2 + 2\gamma_{k,x}^2 \|\nabla_x f(x^k; \theta^*) - \nabla_x f(x^*; \theta^*)\|^2 + 2\gamma_{k,x}^2 \|\nabla_x f(x^*; \theta^*)\|^2 - 2\gamma_{k,x} (f(x^k; \theta^*) - f(x^*; \theta^*)), \end{aligned}$$



where the last inequality follows from the identity  $\|(a - b) + b\|^2 \leq 2\|a - b\|^2 + 2\|b\|^2$ . From the Lipschitz continuity of  $\nabla_x f(x; \theta)$  in  $x$ , the right hand side can be bounded as follows:

$$(2.8) \quad \begin{aligned} & \|x^k - x^*\|^2 + 2\gamma_{k,x}^2 \|\nabla_x f(x^k; \theta^*) - \nabla_x f(x^*; \theta^*)\|^2 + 2\gamma_{k,x}^2 \|\nabla_x f(x^*; \theta^*)\|^2 - 2\gamma_{k,x} (f(x^k; \theta^*) - f(x^*; \theta^*)) \\ & \leq (1 + 2\gamma_{k,x}^2 L_x^2) \|x^k - x^*\|^2 + 2\gamma_{k,x}^2 \|\nabla_x f(x^*; \theta^*)\|^2 - 2\gamma_{k,x} (f(x^k; \theta^*) - f(x^*; \theta^*)). \end{aligned}$$

By the Lipschitz continuity of  $\nabla_x f(x; \theta)$  in  $\theta$  (A1-2),

$$(2.9) \quad \text{Term 2} \leq \gamma_{k,x}^2 L_\theta^2 \|\theta^k - \theta^*\|^2.$$

By adding and subtracting  $\nabla_x f(x^*; \theta^*)$ , and by invoking the Lipschitz continuity of  $\nabla_x f(x; \theta)$  in  $x$  (A1-2) and the triangle inequality, we may derive a bound for **Term 3** as follows:

$$\begin{aligned} \text{Term 3} & \leq 2\gamma_{k,x} \|(x^k - x^*) - \gamma_{k,x} \nabla_x f(x^k; \theta^*)\| \|\nabla_x f(x^k; \theta^k) - \nabla_x f(x^k; \theta^*)\| \\ & \leq 2\gamma_{k,x} \|(x^k - x^*) - \gamma_{k,x} (\nabla_x f(x^k; \theta^*) - \nabla_x f(x^*; \theta^*)) - \gamma_{k,x} \nabla_x f(x^*; \theta^*)\| L_\theta \|\theta^k - \theta^*\| \\ & \leq 2\gamma_{k,x} ((1 + \gamma_{k,x} L_x) \|x^k - x^*\| + \gamma_{k,x} \|\nabla_x f(x^*; \theta^*)\|) L_\theta \|\theta^k - \theta^*\| \\ & = 2\gamma_{k,x} L_\theta \|x^k - x^*\| \|\theta^k - \theta^*\| + 2\gamma_{k,x}^2 L_\theta L_x \|x^k - x^*\| \|\theta^k - \theta^*\| + 2\gamma_{k,x}^2 L_\theta \|\nabla_x f(x^*; \theta^*)\| \|\theta^k - \theta^*\|. \end{aligned}$$

By using the fact that  $2ab \leq a^2 + b^2$ , we have further that

$$(2.10) \quad \begin{aligned} \text{Term 3} & \leq \gamma_{k,x}^{2-\tau} L_\theta^2 \|x^k - x^*\|^2 + \gamma_{k,x}^\tau \|\theta^k - \theta^*\|^2 + \gamma_{k,x}^2 L_\theta L_x \|x^k - x^*\|^2 \\ & \quad + \gamma_{k,x}^2 L_\theta L_x \|\theta^k - \theta^*\|^2 + \gamma_{k,x}^2 L_\theta^2 \|\theta^k - \theta^*\|^2 + \gamma_{k,x}^2 \|\nabla_x f(x^*; \theta^*)\|^2, \end{aligned}$$

where  $\tau \in (0, 1)$  is chosen to satisfy (A2-2). Combining (2.7), (2.8), (2.9) and (2.10), we obtain the following bound on the conditional error.

$$(2.11) \quad \begin{aligned} \mathbb{E}[\|x^{k+1} - x^*\|^2 \mid \mathcal{F}_k] & \leq (1 + \gamma_{k,x}^{2-\tau} L_\theta^2 + \gamma_{k,x}^2 (2L_x^2 + L_\theta L_x)) \|x^k - x^*\|^2 + (\gamma_{k,x}^\tau + \gamma_{k,x}^2 (2L_\theta^2 + L_\theta L_x)) \|\theta^k - \theta^*\|^2 \\ & \quad + 3\gamma_{k,x}^2 \|\nabla_x f(x^*; \theta^*)\|^2 - 2\gamma_{k,x} (f(x^k; \theta^*) - f(x^*; \theta^*)). \end{aligned}$$

From (2.6), we have that

$$(2.12) \quad \mathbb{E}[\|\theta^{k+1} - \theta^*\|^2 \mid \mathcal{F}_k] \leq q_{k,\theta}^2 \|\theta^k - \theta^*\|^2 + \gamma_{k,\theta}^2 \nu_\theta^2,$$

where  $q_{k,\theta} \triangleq \sqrt{1 - 2\gamma_{k,\theta} \mu_\theta + \gamma_{k,\theta}^2 C_\theta^2}$ . Choose  $\beta_k = \frac{\gamma_{k,x}^\tau}{2\gamma_{k,\theta} \mu_\theta}$  by (A2-2). Note that by assumption  $\beta_{k+1} \leq \beta_k$ . By multiplying the left hand side of (2.12) by  $\beta_{k+1}$  and adding to the left hand side of (2.11), we get

$$\begin{aligned} (2.13) \quad & \mathbb{E}[\|x^{k+1} - x^*\|^2 \mid \mathcal{F}_k] + \beta_{k+1} \mathbb{E}[\|\theta^{k+1} - \theta^*\|^2 \mid \mathcal{F}_k] \leq \mathbb{E}[\|x^{k+1} - x^*\|^2 \mid \mathcal{F}_k] + \beta_k \mathbb{E}[\|\theta^{k+1} - \theta^*\|^2 \mid \mathcal{F}_k] \\ & \leq (1 + \gamma_{k,x}^{2-\tau} L_\theta^2 + \gamma_{k,x}^2 (2L_x^2 + L_\theta L_x)) \|x^k - x^*\|^2 + (\beta_k q_{k,\theta}^2 + \gamma_{k,x}^\tau + \gamma_{k,x}^2 (2L_\theta^2 + L_\theta L_x)) \|\theta^k - \theta^*\|^2 \\ & \quad + 3\gamma_{k,x}^2 \|\nabla_x f(x^*; \theta^*)\|^2 + \beta_k \gamma_{k,\theta}^2 \nu_\theta^2 - 2\gamma_{k,x} (f(x^k; \theta^*) - f(x^*; \theta^*)) \\ & \leq (1 + \gamma_{k,x}^{2-\tau} L_\theta^2 + \gamma_{k,x}^2 (2L_x^2 + L_\theta L_x)) \|x^k - x^*\|^2 + \underbrace{\frac{\beta_k q_{k,\theta}^2 + \gamma_{k,x}^\tau + \gamma_{k,x}^2 (2L_\theta^2 + L_\theta L_x)}{\beta_k}}_{\text{Term 4}} \cdot \beta_k \|\theta^k - \theta^*\|^2 \\ & \quad + 3\gamma_{k,x}^2 \|\nabla_x f(x^*; \theta^*)\|^2 + \beta_k \gamma_{k,\theta}^2 \nu_\theta^2 - 2\gamma_{k,x} (f(x^k; \theta^*) - f(x^*; \theta^*)). \end{aligned}$$

Term 4 on the right hand side of (2.13) can be further expanded as

$$\begin{aligned}
 (2.14) \quad \frac{\beta_k q_{k,\theta}^2 + \gamma_{k,x}^\tau + \gamma_{k,x}^2(2L_\theta^2 + L_\theta L_x)}{\beta_k} &= q_{k,\theta}^2 + \frac{\gamma_{k,x}^\tau + \gamma_{k,x}^2(2L_\theta^2 + L_\theta L_x)}{\beta_k} \\
 &= 1 - 2\gamma_{k,\theta}\mu_\theta + \gamma_{k,\theta}^2 C_\theta^2 + \frac{\gamma_{k,x}^\tau}{\beta_k} + \frac{\gamma_{k,x}^2(2L_\theta^2 + L_\theta L_x)}{\beta_k} \\
 &= 1 + \gamma_{k,\theta}^2 C_\theta^2 + 2\gamma_{k,\theta}\gamma_{k,x}^{2-\tau}\mu_\theta(2L_\theta^2 + L_\theta L_x).
 \end{aligned}$$

Combining (2.13) and (2.14), we get

$$\begin{aligned}
 &\mathbb{E}[\|x^{k+1} - x^*\|^2 \mid \mathcal{F}_k] + \beta_{k+1}\mathbb{E}[\|\theta^{k+1} - \theta^*\|^2 \mid \mathcal{F}_k] \\
 &\leq (1 + \gamma_{k,x}^{2-\tau}L_\theta^2 + \gamma_{k,x}^2(2L_\theta^2 + L_\theta L_x))\|x^k - x^*\|^2 + (1 + \gamma_{k,\theta}^2 C_\theta^2 + 2\gamma_{k,\theta}\gamma_{k,x}^{2-\tau}\mu_\theta(2L_\theta^2 + L_\theta L_x))\beta_k\|\theta^k - \theta^*\|^2 \\
 &\quad + 3\gamma_{k,x}^2\|\nabla_x f(x^*; \theta^*)\|^2 + \beta_k\gamma_{k,\theta}^2\nu_\theta^2 - 2\gamma_{k,x}(f(x^k; \theta^*) - f(x^*; \theta^*)) \\
 &\leq (1 + \gamma_{k,\theta}^2 C_\theta^2 + 2\gamma_{k,\theta}\gamma_{k,x}^{2-\tau}\mu_\theta(2L_\theta^2 + L_\theta L_x))(\|x^k - x^*\|^2 + \beta_k\|\theta^k - \theta^*\|^2) + (\gamma_{k,x}^{2-\tau}L_\theta^2 + \gamma_{k,x}^2(2L_\theta^2 + L_\theta L_x))\|x^k - x^*\|^2 \\
 &\quad + 3\gamma_{k,x}^2\|\nabla_x f(x^*; \theta^*)\|^2 + \beta_k\gamma_{k,\theta}^2\nu_\theta^2 - 2\gamma_{k,x}(f(x^k; \theta^*) - f(x^*; \theta^*)).
 \end{aligned}$$

We define the following:

$$\begin{aligned}
 u_k &\triangleq \gamma_{k,\theta}^2 C_\theta^2 + 2\gamma_{k,\theta}\gamma_{k,x}^{2-\tau}\mu_\theta(2L_\theta^2 + L_\theta L_x), \sigma_k \triangleq 2\gamma_{k,x}(f(x^k; \theta^*) - f(x^*; \theta^*)), \\
 \text{and } \rho_k &\triangleq (\gamma_{k,x}^{2-\tau}L_\theta^2 + \gamma_{k,x}^2(2L_\theta^2 + L_\theta L_x))\|x^k - x^*\|^2 + 3\gamma_{k,x}^2\|\nabla_x f(x^*; \theta^*)\|^2 + \beta_k\gamma_{k,\theta}^2\nu_\theta^2.
 \end{aligned}$$

Then, we have

$$\mathbb{E}[\|x^{k+1} - x^*\|^2 \mid \mathcal{F}_k] + \beta_{k+1}\mathbb{E}[\|\theta^{k+1} - \theta^*\|^2 \mid \mathcal{F}_k] \leq (1 + u_k)(\|x^k - x^*\|^2 + \beta_k\|\theta^k - \theta^*\|^2) + \rho_k - \sigma_k.$$

By boundedness of  $X$  and (A2-2), we have that  $\sum_{k=0}^\infty u_k < \infty$  and  $\sum_{k=0}^\infty \rho_k < \infty$ . So, by Lemma 2.3 we get that there exists a random variable  $V$  such that  $\|x^k - x^*\|^2 + \beta_k\|\theta^k - \theta^*\|^2 \rightarrow V$  in an almost sure sense as  $k \rightarrow \infty$  and  $\sum_{k=0}^\infty \sigma_k = \sum_{k=0}^\infty 2\gamma_{k,x}(f(x^k; \theta^*) - f(x^*; \theta^*)) < \infty$ .

By (A2-2), Lemma 2.2 and (2.12), we can get that  $\|\theta^k - \theta^*\| \rightarrow 0$  a.s. as  $k \rightarrow \infty$ . Thus, it follows that  $\|x^k - x^*\| \rightarrow V$  a.s. as  $k \rightarrow \infty$ . Since  $\sum_{k=0}^\infty \gamma_{k,x} = \infty$ , we get  $\liminf_{k \rightarrow \infty} f(x^k; \theta^*) = f(x^*; \theta^*)$  a.s. as  $k \rightarrow \infty$ . Since the set  $X$  is closed, all accumulation points of  $\{x^k\}$  lie in  $X$ . Furthermore, since  $f(x^k; \theta^*) \rightarrow f(x^*; \theta^*)$  along a subsequence a.s., by continuity of  $f$  it follows that  $\{x^k\}$  has a subsequence converging a.s. to some point in  $X$ , say  $\tilde{x}$ , which satisfies  $f(\tilde{x}; \theta^*) = f(x^*; \theta^*)$ . That means  $\tilde{x}$  is some random point in  $X^*$ . Moreover, since  $\|x^k - x^*\|$  is convergent for any  $x^* \in X^*$  a.s., the entire sequence  $\{x^k\}$  converges to some random point in  $X^*$  a.s.  $\square$

**2.3. Diminishing and constant steplength rate analysis.** While the previous section focused on the almost sure convergence of the prescribed learning and computational schemes, a natural question is whether one can develop rate statements. We begin with an examination of the global rate of convergence and show that  $\mathcal{O}(1/K)$  rate estimate is derived for an upper bound on the mean-squared error in the solution  $x_K$  when  $f(\bullet; \theta^*)$  is strongly convex in  $(\bullet)$  and  $K$  represents the number of steps, consistent with the result obtained for stochastic approximation (cf. [40, 45]). In addition, it is seen that when the function loses strong convexity, an analogous rate estimate is available by using averaging, akin to an approach first employed in [42], where longer stepsizes were suggested with consequent averaging of the obtained iterates.

**PROPOSITION 2 (Rate estimates for strongly convex  $f$ ).** Suppose (A1-1) and (A3) hold. Suppose  $\gamma_{x,k} = \lambda_x/k$  and  $\gamma_{\theta,k} = \lambda_\theta/k$  with  $\lambda_x > 1/\mu_x$  and  $\lambda_\theta > 1/(2\mu_\theta)$ . Let  $\mathbb{E}[\|\nabla_x f(x^k; \theta^k) + w^k\|^2] \leq M^2$  and  $\mathbb{E}[\|\nabla_\theta g(\theta^k) + v^k\|^2] \leq M_\theta^2$  for all  $x^k \in X$  and  $\theta^k \in \Theta$ . Let  $\{x^k, \theta^k\}$  be computed via Algorithm 1. Then, the following hold after  $K$  iterations:

$$\mathbb{E}[\|\theta^K - \theta^*\|^2] \leq \frac{Q_\theta(\lambda_\theta)}{K} \text{ and } \mathbb{E}[\|x^K - x^*\|^2] \leq \frac{Q_x(\lambda_x)}{K},$$

where  $Q_\theta(\lambda_\theta) \triangleq \max \{ \lambda_\theta^2 M_\theta^2 (2\mu_\theta \lambda_\theta - 1)^{-1}, \mathbb{E}[\|\theta^1 - \theta^*\|^2] \}$ ,

$$Q_x(\lambda_x) \triangleq \max \left\{ \lambda_x^2 \widetilde{M}^2 (\mu_x \lambda_x - 1)^{-1}, \mathbb{E}[\|x^1 - x^*\|^2] \right\}, \text{ and } \widetilde{M} \triangleq \sqrt{M^2 + \frac{L_\theta^2 Q_\theta(\lambda_\theta)}{\mu_x \lambda_x}}.$$

*Proof.* Suppose  $A_k \triangleq \frac{1}{2}\|x^k - x^*\|^2$  and  $a_k \triangleq \mathbb{E}[A_k]$ . Then,  $A_{k+1}$  may be bounded as follows by using the non-expansivity of the Euclidean projector:

$$\begin{aligned} A_{k+1} &= \frac{1}{2}\|x^{k+1} - x^*\|^2 = \frac{1}{2}\|\Pi_X(x^k - \gamma_{x,k}(\nabla_x f(x^k; \theta^k) + w^k)) - \Pi_X(x^*)\|^2 \\ (2.15) \quad &\leq \frac{1}{2}\|x^k - x^* - \gamma_{x,k}(\nabla_x f(x^k; \theta^k) + w^k)\|^2 \\ &= A_k + \frac{1}{2}\gamma_{x,k}^2 \|\nabla_x f(x^k; \theta^k) + w^k\|^2 - \gamma_{x,k}(x^k - x^*)^T (\nabla_x f(x^k; \theta^k) + w^k). \end{aligned}$$

Note that  $\mathbb{E}[(x^k - x^*)^T w^k] = \mathbb{E}[\mathbb{E}[(x^k - x^*)^T w^k | \mathcal{F}_k]] = \mathbb{E}[(x^k - x^*)^T \mathbb{E}[w^k | \mathcal{F}_k]] = 0$ . By taking expectations on both sides of (2.15) and by invoking the bounds  $\mathbb{E}[\|\nabla_x f(x^k; \theta^k) + w^k\|^2] \leq M^2$  and  $\mathbb{E}[\|\nabla_\theta g(\theta^k) + v^k\|^2] \leq M_\theta^2$ , it follows that

$$(2.16) \quad a_{k+1} \leq a_k + \frac{1}{2}\gamma_{x,k}^2 M^2 - \gamma_{x,k} \mathbb{E}[(x^k - x^*)^T \nabla_x f(x^k; \theta^k)].$$

But  $f(x; \theta)$  is strongly convex in  $x$  with constant  $\mu_x$  for every  $\theta \in \Theta$ , leading to the following expression:

$$\begin{aligned} \mathbb{E}[(x^k - x^*)^T \nabla_x f(x^k; \theta^k)] &= \mathbb{E}[(x^k - x^*)^T (\nabla_x f(x^k; \theta^k) - \nabla_x f(x^*; \theta^k))] \\ (2.17) \quad &+ \mathbb{E}[(x^k - x^*)^T (\nabla_x f(x^*; \theta^k) - \nabla_x f(x^*; \theta^*))] + \mathbb{E}[(x^k - x^*)^T \nabla_x f(x^*; \theta^*)] \\ &\geq \mu_x \mathbb{E}[\|x^k - x^*\|^2] + \mathbb{E}[(x^k - x^*)^T (\nabla_x f(x^*; \theta^k) - \nabla_x f(x^*; \theta^*))]. \end{aligned}$$

Combining (2.16) and (2.17), we get

$$\begin{aligned} a_{k+1} &\leq (1 - 2\gamma_{x,k}\mu_x)a_k + \frac{1}{2}\gamma_{x,k}^2 M^2 - \gamma_{x,k} \mathbb{E}[(x^k - x^*)^T (\nabla_x f(x^*; \theta^k) - \nabla_x f(x^*; \theta^*))] \\ &\leq (1 - 2\gamma_{x,k}\mu_x)a_k + \frac{1}{2}\gamma_{x,k}^2 M^2 + \frac{1}{2}\gamma_{x,k}\mu_x \mathbb{E}[\|x^k - x^*\|^2] + \frac{1}{2}\frac{\gamma_{x,k}}{\mu_x} \mathbb{E}[\|\nabla_x f(x^*; \theta^k) - \nabla_x f(x^*; \theta^*)\|^2] \\ (2.18) \quad &\leq (1 - \gamma_{x,k}\mu_x)a_k + \frac{1}{2}\gamma_{x,k}^2 M^2 + \frac{1}{2}\frac{\gamma_{x,k}}{\mu_x} L_\theta^2 \mathbb{E}[\|\theta^k - \theta^*\|^2]. \end{aligned}$$

Suppose  $\gamma_{\theta,k} = \lambda_\theta/k$ . Since the function  $g(\theta)$  is strongly convex, we can use the standard rate estimate (cf. inequality (5.292) in [45]) to get the following

$$(2.19) \quad \mathbb{E}[\|\theta^k - \theta^*\|^2] \leq \frac{Q_\theta(\lambda_\theta)}{k},$$

where  $Q_\theta(\lambda_\theta) \triangleq \max \{ \lambda_\theta^2 M_\theta^2 (2\mu_\theta \lambda_\theta - 1)^{-1}, \mathbb{E}[\|\theta^1 - \theta^*\|^2] \}$  with  $\lambda_\theta > 1/(2\mu_\theta)$ . Suppose  $\gamma_{x,k} = \lambda_x/k$ , allowing us to claim the following:

$$a_{k+1} \leq \left(1 - \frac{\mu_x \lambda_x}{k}\right) a_k + \frac{1}{2} \frac{\lambda_x^2 M^2}{k^2} + \frac{1}{2} \frac{\lambda_x L_\theta^2 Q_\theta(\lambda_\theta)}{\mu_x k^2} = \left(1 - \frac{\mu_x \lambda_x}{k}\right) a_k + \frac{1}{2} \frac{\lambda_x^2 \widetilde{M}^2}{k^2},$$

where  $\widetilde{M} \triangleq \sqrt{M^2 + \frac{L_\theta^2 Q_\theta(\lambda_\theta)}{\mu_x \lambda_x}}$ . By assuming that  $\lambda_x > 1/\mu_x$ , the result follows by observing that

$$\mathbb{E}[\|x^k - x^*\|^2] \leq \frac{Q_x(\lambda_x)}{k},$$

where  $Q_x(\lambda_x) \triangleq \max \left\{ \lambda_x^2 \widetilde{M}^2 (\mu_x \lambda_x - 1)^{-1}, \mathbb{E}[\|x^1 - x^*\|^2] \right\}$ .  $\square$

**Remark:** Notice that here we assume that  $f$  and  $g$  are both smooth and strongly convex. A more general framework is that of composite objectives where the objective is a sum of nonsmooth and smooth stochastic components. Lan [35] proposed the accelerated stochastic approximation (AC-SA) algorithm for solving stochastic composite optimization (SCO) problems and proved that it achieves the optimal rate. In related work, Ghadimi and Lan [19, 20] propose a multi-stage AC-SA algorithm, which possesses an optimal rate of convergence for solving strongly convex SCO problems in terms of the dependence on different problem parameters. While this problem class is beyond the current scope, this approach may aid in refinement of the constants in the Proposition 2 in some regimes.

A shortcoming of the previous result is the need for strong convexity of  $f(x, \theta)$  in  $x$  for every  $\theta \in \Theta$ . In our next result, we weaken this requirement and allow for a merely convex  $f$ , extending the optimal constant stepsize result in [45]. Specifically, given a prescribed number of iterations, say  $K$ , the optimal “constant stepsize” derives the error minimizing steplength; in other words,  $\gamma_k = \gamma$  for  $1 \leq k \leq K$ . This is in contrast with the constant stepsize result presented in Proposition 3, where  $\gamma_k = \gamma$  for all  $k$ . steps. The following Lipschitzian assumption is imposed on the function  $f(x; \theta)$ .

ASSUMPTION 6 (A6). *Suppose the following holds in addition to (A1-2).*

(i) *For every  $x \in X$ ,  $f(x; \theta)$  is Lipschitz continuous in  $\theta$  with constant  $D_\theta$ .*

**THEOREM 2 (Rate estimates under convexity of  $f$ ).** *Suppose (A3) and (A6) hold. Suppose  $\mathbb{E}[\|x^k - x^*\|^2] \leq M_x^2$ ,  $\mathbb{E}[\|\nabla_x f(x^k; \theta^k) + w^k\|^2] \leq M^2$  and  $\mathbb{E}[\|\nabla_\theta g(\theta^k) + v^k\|^2] \leq M_\theta^2$  for all  $x^k \in X$  and  $\theta^k \in \Theta$ . Let  $\{x^k, \theta^k\}$  be computed via Algorithm 1. For  $1 \leq i, t \leq k$ , we define  $v_t \triangleq \frac{\gamma_{x,t}}{\sum_{s=i}^k \gamma_{x,s}}$ ,  $\tilde{x}_{i,k} \triangleq \sum_{t=i}^k v_t x^t$  and  $D_X \triangleq \max_{x \in X} \|x - x^1\|$ . Suppose for  $1 \leq t \leq K$ ,  $\gamma_x$  is defined as follows:*

$$\gamma_x = \sqrt{\frac{4D_X^2 + L_\theta^2 Q_\theta(\lambda_\theta)(1 + \ln K)}{(M^2 + M_x^2)K}},$$

*where  $Q_\theta(\lambda_\theta) \triangleq \max \{ \lambda_\theta^2 M_\theta^2 (2\mu_\theta \lambda_\theta - 1)^{-1}, \mathbb{E}[\|\theta^1 - \theta^*\|^2] \}$ , and  $\gamma_{\theta,k} = \lambda_\theta/K$  with  $\lambda_\theta > 1/(2\mu_\theta)$ . Then the following holds for  $1 \leq i \leq K$ :*

$$|\mathbb{E}[f(\tilde{x}_{i,K}; \theta^K) - f(x^*; \theta^*)]| \leq \frac{\sqrt{Q_\theta(\lambda_\theta)} D_\theta + C_{i,K} \sqrt{B_K}}{\sqrt{K}},$$

*where  $C_{i,K} = \frac{K}{K-i+1}$  and  $B_K = (4D_X^2 + L_\theta^2 Q_\theta(\lambda_\theta)(1 + \ln K))(M^2 + M_x^2)$ .*

*Proof.* By using the same notation in Proposition 2, we have from (2.16) that

$$\begin{aligned}
 a_{k+1} &\leq a_k + \frac{1}{2}\gamma_{x,k}^2 M^2 - \gamma_{x,k} \mathbb{E}[(x^k - x^*)^T \nabla_x f(x^k; \theta^k)] \\
 (2.20) \quad &\leq a_k + \frac{1}{2}\gamma_{x,k}^2 M^2 - \gamma_{x,k} \mathbb{E}[(x^k - x^*)^T \nabla_x f(x^k; \theta^*)] - \gamma_{x,k} \mathbb{E}[(x^k - x^*)^T (\nabla_x f(x^k; \theta^k) - \nabla_x f(x^k; \theta^*))].
 \end{aligned}$$

Note that  $f(x; \theta)$  is convex in  $x$  for every  $\theta \in \Theta$ , allowing us to leverage the gradient inequality.

$$(2.21) \quad \mathbb{E}[(x^k - x^*)^T \nabla_x f(x^k; \theta^*)] \geq \mathbb{E}[f(x^k; \theta^*) - f(x^*; \theta^*)].$$

Combining (2.20) and (2.21), we obtain the following:

$$a_{k+1} \leq a_k + \frac{1}{2}\gamma_{x,k}^2 M^2 - \gamma_{x,k} \mathbb{E}[f(x^k; \theta^*) - f(x^*; \theta^*)] - \gamma_{x,k} \mathbb{E}[(x^k - x^*)^T (\nabla_x f(x^k; \theta^k) - \nabla_x f(x^k; \theta^*))].$$

This allows for constructing the following bounds:

$$\begin{aligned}
 \gamma_{x,k} \mathbb{E}[f(x^k; \theta^*) - f(x^*; \theta^*)] &\leq a_k - a_{k+1} + \frac{1}{2}\gamma_{x,k}^2 M^2 - \gamma_{x,k} \mathbb{E}[(x^k - x^*)^T (\nabla_x f(x^k; \theta^k) - \nabla_x f(x^k; \theta^*))] \\
 &\leq a_k - a_{k+1} + \frac{1}{2}\gamma_{x,k}^2 M^2 + \frac{1}{2}\gamma_{x,k}^2 \mathbb{E}[\|x^k - x^*\|^2] + \frac{1}{2}\mathbb{E}[\|\nabla_x f(x^k; \theta^k) - \nabla_x f(x^k; \theta^*)\|^2] \\
 &\leq a_k - a_{k+1} + \frac{1}{2}\gamma_{x,k}^2 M^2 + \frac{1}{2}\gamma_{x,k}^2 M_x^2 + \frac{1}{2}L_\theta^2 \mathbb{E}[\|\theta^k - \theta^*\|^2] \\
 (2.22) \quad &\leq a_k - a_{k+1} + \frac{1}{2}\gamma_{x,k}^2 (M^2 + M_x^2) + \frac{1}{2} \frac{L_\theta^2 Q_\theta(\lambda_\theta)}{k},
 \end{aligned}$$

where the second inequality follows from the fact that  $2ab \leq a^2 + b^2$ , the third inequality follows from the boundedness of  $\mathbb{E}[\|x^k - x^*\|^2]$  and Lipschitz continuity of  $\nabla_x f(x; \theta)$  in  $\theta$ , and the last inequality follows from (2.19). As a result, for  $1 \leq i \leq k$ , we have the following:

$$\begin{aligned}
 \sum_{t=i}^k \gamma_{x,t} \mathbb{E}[f(x^t; \theta^*) - f(x^*; \theta^*)] &\leq \sum_{t=i}^k (a_t - a_{t+1}) + \frac{1}{2} \sum_{t=i}^k \gamma_{x,t}^2 (M^2 + M_x^2) + \frac{1}{2} \sum_{t=i}^k \frac{L_\theta^2 Q_\theta(\lambda_\theta)}{t} \\
 &\leq a_i + \frac{1}{2} \sum_{t=i}^k \gamma_{x,t}^2 (M^2 + M_x^2) + \frac{1}{2} \sum_{t=i}^k \frac{L_\theta^2 Q_\theta(\lambda_\theta)}{t} \\
 (2.23) \quad &\leq a_i + \frac{1}{2} \sum_{t=i}^k \gamma_{x,t}^2 (M^2 + M_x^2) + \frac{1}{2} L_\theta^2 Q_\theta(\lambda_\theta) (1 + \ln k).
 \end{aligned}$$

Next, we define  $v_t \triangleq \frac{\gamma_{x,t}}{\sum_{s=i}^k \gamma_{x,s}}$  and  $D_X \triangleq \max_{x \in X} \|x - x^1\|$ . The following holds invoking these definitions:

$$(2.24) \quad \mathbb{E} \left[ \sum_{t=i}^k v_t f(x^t; \theta^*) - f(x^*; \theta^*) \right] \leq \frac{a_i + \frac{1}{2} \sum_{t=i}^k \gamma_{x,t}^2 (M^2 + M_x^2) + \frac{1}{2} L_\theta^2 Q_\theta(\lambda_\theta) (1 + \ln k)}{\sum_{t=i}^k \gamma_{x,t}}.$$

Next, we consider points given by  $\tilde{x}_{i,k} \triangleq \sum_{t=i}^k v_t x^t$ . By convexity of  $X$ , we have that  $\tilde{x}_{i,k} \in X$  and by the convexity of  $f(x; \theta^*)$  in  $x$ , we have  $f(\tilde{x}_{i,k}; \theta^*) \leq \sum_{t=i}^k v_t f(x^t; \theta^*)$ . From (2.24) and by noting that  $a_1 \leq \frac{1}{2} D_X^2$  and  $a_i \leq 2D_X^2$  for  $i > 1$ , we obtain the following for  $1 \leq i \leq k$

$$(2.25) \quad \mathbb{E}[f(\tilde{x}_{i,k}; \theta^*) - f(x^*; \theta^*)] \leq \frac{4D_X^2 + \sum_{t=i}^k \gamma_{x,t}^2 (M^2 + M_x^2) + L_\theta^2 Q_\theta(\lambda_\theta) (1 + \ln k)}{2 \sum_{t=i}^k \gamma_{x,t}}.$$

Suppose  $\gamma_{x,t} = \gamma_x$  for  $t = 1, \dots, k$ . Then, it follows that

$$(2.26) \quad \mathbb{E}[f(\tilde{x}_{1,k}; \theta^*) - f(x^*; \theta^*)] \leq \frac{4D_X^2 + k\gamma_x^2(M^2 + M_x^2) + L_\theta^2 Q_\theta(\lambda_\theta)(1 + \ln k)}{2k\gamma_x}.$$

By minimizing the right hand side in  $\gamma_x > 0$ , we obtain that

$$\gamma_x = \sqrt{\frac{4D_X^2 + L_\theta^2 Q_\theta(\lambda_\theta)(1 + \ln k)}{(M^2 + M_x^2)k}}.$$

This implies the following bound:

$$\mathbb{E}[f(\tilde{x}_{1,k}; \theta^*) - f(x^*; \theta^*)] \leq \sqrt{\frac{B_k}{k}},$$

where  $B_k \triangleq (4D_X^2 + L_\theta^2 Q_\theta(\lambda_\theta)(1 + \ln k))(M^2 + M_x^2)$ . Next, we can also claim that for  $1 \leq i \leq k$ ,

$$(2.27) \quad \mathbb{E}[f(\tilde{x}_{i,k}; \theta^*) - f(x^*; \theta^*)] \leq C_{i,k} \sqrt{\frac{B_k}{k}},$$

where  $C_{i,k} = \frac{k}{k-i+1}$ . Thus, by employing (2.19), (2.27) and the Lipschitz continuity of  $f(x; \theta)$  in  $\theta$ , we have the required result:

$$\begin{aligned} |\mathbb{E}[f(\tilde{x}_{i,k}; \theta^k) - f(x^*; \theta^*)]| &\leq |\mathbb{E}[f(\tilde{x}_{i,k}; \theta^k) - f(\tilde{x}_{i,k}; \theta^*)]| + |\mathbb{E}[f(\tilde{x}_{i,k}; \theta^*) - f(x^*; \theta^*)]| \\ &\leq D_\theta \mathbb{E}[|\theta^k - \theta^*|] + \mathbb{E}[f(\tilde{x}_{i,k}; \theta^*) - f(x^*; \theta^*)] \\ &\leq \frac{\sqrt{Q_\theta(\lambda_\theta)} D_\theta}{\sqrt{k}} + \mathbb{E}[f(\tilde{x}_{i,k}; \theta^*) - f(x^*; \theta^*)] \leq \frac{\sqrt{Q_\theta(\lambda_\theta)} D_\theta + C_{i,k} \sqrt{B_k}}{\sqrt{k}}. \end{aligned}$$

□

**Remark:** In effect, in the context of learning and optimization, the averaging approach leads to a complexity bound given loosely by

$$\mathcal{O} \left( \frac{a_\theta}{\sqrt{K}} + \underbrace{\frac{b + c_\theta \sqrt{\ln(K)}}{\sqrt{K}}}_{\text{degradation from learning}} \right),$$

where  $a_\theta, b, c_\theta$  are suitably defined. If  $\theta^*$  is available, then  $a_\theta, c_\theta = 0$ , leading to the standard bound of  $\mathcal{O}(1/\sqrt{K})$ . While it is not surprising that the requirement to learn  $\theta^*$  imposes a degradation, it appears that this degradation is not severe. However, by changing the averaging window, this degradation disappears from a rate standpoint. Specifically, the next result is a corollary of Theorem 2 and uses a modified averaging window, as seen in [40].

**COROLLARY 1 (Rate estimates under convexity of  $f$ ).** Suppose (A3) and (A6) hold. Suppose  $\mathbb{E}[\|x^k - x^*\|^2] \leq M_x^2$ ,  $\mathbb{E}[\|\nabla_x f(x^k; \theta^k) + w^k\|^2] \leq M^2$  and  $\mathbb{E}[\|\nabla_\theta g(\theta^k) + v^k\|^2] \leq M_\theta^2$  for all  $x^k \in X$  and  $\theta^k \in \Theta$ . Let  $\{x^k, \theta^k\}$  be computed via Algorithm 1. Let  $k$  be a positive even number. For  $k/2 \leq t \leq k$ , we define  $v_t \triangleq \frac{\gamma_{x,t}}{\sum_{s=k/2}^k \gamma_{x,s}}$ ,  $\tilde{x}_{k/2,k} \triangleq \sum_{t=k/2}^k v_t x^t$  and  $D_X \triangleq \max_{x \in X} \|x - x^1\|$ . Suppose for  $1 \leq t \leq K$ ,  $\gamma_x$  is defined as follows:

$$\gamma_x = \sqrt{\frac{4D_X^2 + L_\theta^2 Q_\theta(\lambda_\theta)(1 + \ln 2)}{(M^2 + M_x^2)k}},$$

where  $Q_\theta(\lambda_\theta) \triangleq \max\{\lambda_\theta^2 M_\theta^2 (2\mu_\theta \lambda_\theta - 1)^{-1}, \mathbb{E}[\|\theta^1 - \theta^*\|^2]\}$ , and  $\gamma_{\theta,k} = \lambda_\theta/K$  with  $\lambda_\theta > 1/(2\mu_\theta)$ . Then the following holds:

$$|\mathbb{E}[f(\tilde{x}_{K/2,K}; \theta^K) - f(x^*; \theta^*)]| \leq \frac{\sqrt{Q_\theta(\lambda_\theta)} D_\theta + 2\sqrt{B}}{\sqrt{K}},$$

where  $B \triangleq (4D_X^2 + L_\theta^2 Q_\theta(\lambda_\theta)(1 + \ln 2))(M^2 + M_x^2)$ .

*Proof.* When  $i = k/2$  where  $k$  is a positive even number, the second inequality of (2.23) becomes

$$\begin{aligned} \sum_{t=k/2}^k \gamma_{x,t} \mathbb{E}[f(x^t; \theta^*) - f(x^*; \theta^*)] &\leq a_{k/2} + \frac{1}{2} \sum_{t=k/2}^k \gamma_{x,t}^2 (M^2 + M_x^2) + \frac{1}{2} \sum_{t=k/2}^k \frac{L_\theta^2 Q_\theta(\lambda_\theta)}{t} \\ &\leq a_{k/2} + \frac{1}{2} \sum_{t=k/2}^k \gamma_{x,t}^2 (M^2 + M_x^2) + \frac{1}{2} L_\theta^2 Q_\theta(\lambda_\theta) \left[ \sum_{t=1}^k \frac{1}{t} - \sum_{t=k/2-1}^k \frac{1}{t} \right] \\ &\leq a_{k/2} + \frac{1}{2} \sum_{t=k/2}^k \gamma_{x,t}^2 (M^2 + M_x^2) + \frac{1}{2} L_\theta^2 Q_\theta(\lambda_\theta) [1 + \ln(k) - \ln(k/2)] \\ &\leq a_{k/2} + \frac{1}{2} \sum_{t=k/2}^k \gamma_{x,t}^2 (M^2 + M_x^2) + \frac{1}{2} L_\theta^2 Q_\theta(\lambda_\theta) (1 + \ln 2). \end{aligned} \tag{2.28}$$

Then, (2.26) becomes

$$\mathbb{E}[f(\tilde{x}_{1,k}; \theta^*) - f(x^*; \theta^*)] \leq \frac{4D_X^2 + k\gamma_x^2 (M^2 + M_x^2) + L_\theta^2 Q_\theta(\lambda_\theta)(1 + \ln 2)}{2k\gamma_x}.$$

By minimizing the right hand side in  $\gamma_x > 0$ , we obtain that

$$\gamma_x = \sqrt{\frac{4D_X^2 + L_\theta^2 Q_\theta(\lambda_\theta)(1 + \ln 2)}{(M^2 + M_x^2)k}}.$$

This implies the following bound:

$$\mathbb{E}[f(\tilde{x}_{1,k}; \theta^*) - f(x^*; \theta^*)] \leq \sqrt{\frac{B}{k}},$$

where  $B \triangleq (4D_X^2 + L_\theta^2 Q_\theta(\lambda_\theta)(1 + \ln 2))(M^2 + M_x^2)$ . Next, we can also claim that,

$$\mathbb{E}[f(\tilde{x}_{k/2,k}; \theta^*) - f(x^*; \theta^*)] \leq C_k \sqrt{\frac{B}{k}},$$

where  $C_k = \frac{k}{k-k/2+1} \leq 2$ . Thus, we have the required result:

$$|\mathbb{E}[f(\tilde{x}_{k/2,k}; \theta^k) - f(x^*; \theta^*)]| \leq \frac{\sqrt{Q_\theta(\lambda_\theta)} D_\theta + 2\sqrt{B}}{\sqrt{k}}.$$

□

We now present a constant steplength error bound where the steplength is fixed over the entire algorithm. As mentioned before, this differs from Theorem 2 in that the number of iterations is not fixed. Constant steplength statements are particularly relevant in networked regimes where the coordination of changing steplength sequences across a collection of agents may prove complicated.

**PROPOSITION 3 (Constant steplength error bound).** *Suppose (A3) holds. Suppose  $\gamma_{\theta,k} := \gamma_\theta$  and  $\gamma_{x,k} := \gamma_x$ . Suppose  $\mathbb{E}[\|x^k - x^*\|^2] \leq M_x^2$  and  $\mathbb{E}[\|\nabla_x f(x^k; \theta^k) + w^k\|^2] \leq M^2$  for all  $x^k \in X$ . Suppose  $A_k \triangleq \frac{1}{2}\|x^k - x^*\|^2$  and  $a_k \triangleq \mathbb{E}[A_k]$ . Let  $\{x^k, \theta^k\}$  be computed via Algorithm 1.*

(i) *Suppose (A1-1) holds. Then, the following holds:*

$$\limsup_{k \rightarrow \infty} a_k \leq \frac{1}{2\mu_x} \gamma_x M^2 + \frac{1}{2\mu_x^2} \frac{\gamma_\theta \nu_\theta^2 L_\theta^2}{2\mu_\theta - \gamma_\theta C_\theta^2};$$

(ii) *Suppose (A1-2) and (A6) hold and  $0 < \tau < 1$ . Then, the following holds:*

$$\limsup_{k \rightarrow \infty} |\mathbb{E}[f(x^k; \theta^k) - f(x^*; \theta^*)]| \leq \frac{1}{2} \gamma_x M^2 + \frac{1}{2} \gamma_x^{1-\tau} M_x^2 + \frac{1}{2} \gamma_x^{\tau-1} L_\theta^2 \frac{\gamma_\theta \nu_\theta^2}{2\mu_\theta - \gamma_\theta C_\theta^2} + D_\theta \sqrt{\frac{\gamma_\theta \nu_\theta^2}{2\mu_\theta - \gamma_\theta C_\theta^2}}.$$

*Proof.* By (2.6), we get the following:

$$\mathbb{E}[\|\theta^{k+1} - \theta^*\|^2 \mid \mathcal{F}_k] \leq q_{k,\theta}^2 \|\theta^k - \theta^*\|^2 + \gamma_{k,\theta}^2 \nu_\theta^2,$$

where  $q_{k,\theta} \triangleq \sqrt{1 - 2\gamma_{k,\theta}\mu_\theta + \gamma_{k,\theta}^2 C_\theta^2}$ . Suppose  $\gamma_{\theta,k} := \gamma_\theta$  is chosen such that  $(1 - q_\theta) < 1$  where  $q_{\theta,k} := q_\theta$ . By taking the expectation and limit supremum on both sides, we have

$$\limsup_{k \rightarrow \infty} \mathbb{E}[\|\theta^{k+1} - \theta^*\|^2] \leq q_\theta^2 \limsup_{k \rightarrow \infty} \mathbb{E}[\|\theta^k - \theta^*\|^2] + \gamma_\theta^2 \nu_\theta^2,$$

or,

$$(2.29) \quad \limsup_{k \rightarrow \infty} \mathbb{E}[\|\theta^k - \theta^*\|^2] \leq \frac{\gamma_\theta \nu_\theta^2}{2\mu_\theta - \gamma_\theta C_\theta^2}.$$

(i)  $f$  is strongly convex: From (2.18), for  $\gamma_{x,k} := \gamma_x$  where  $\gamma_x$  is sufficiently small, we have the following:

$$a_{k+1} \leq (1 - \gamma_x \mu_x) a_k + \frac{1}{2} \gamma_x^2 M^2 + \frac{1}{2} \frac{\gamma_x}{\mu_x} L_\theta^2 \mathbb{E}[\|\theta^k - \theta^*\|^2].$$

It follows that

$$\begin{aligned} \limsup_{k \rightarrow \infty} a_{k+1} &\leq (1 - \gamma_x \mu_x) \limsup_{k \rightarrow \infty} a_k + \frac{1}{2} \gamma_x^2 M^2 + \frac{1}{2} \frac{\gamma_x}{\mu_x} L_\theta^2 \limsup_{k \rightarrow \infty} \mathbb{E}[\|\theta^k - \theta^*\|^2] \\ &\leq (1 - \gamma_x \mu_x) \limsup_{k \rightarrow \infty} a_k + \frac{1}{2} \gamma_x^2 M^2 + \frac{1}{2} \frac{\gamma_x}{\mu_x} L_\theta^2 \frac{\gamma_\theta \nu_\theta^2}{2\mu_\theta - \gamma_\theta C_\theta^2}. \end{aligned}$$



It follows that

$$\limsup_{k \rightarrow \infty} a_k \leq \frac{1}{2\mu_x} \gamma_x M^2 + \frac{1}{2} \frac{1}{\mu_x^2} L_\theta^2 \frac{\gamma_\theta \nu_\theta^2}{2\mu_\theta - \gamma_\theta C_\theta^2}.$$

(ii)  $f$  is convex: From (2.22), for  $\gamma_{x,k} := \gamma_x$ , we have the following:

$$\begin{aligned} \gamma_x \mathbb{E}[f(x^k; \theta^*) - f(x^*; \theta^*)] &\leq a_k - a_{k+1} + \frac{1}{2} \gamma_x^2 M^2 - \gamma_x \mathbb{E}[(x^k - x^*)^T (\nabla_x f(x^k; \theta^k) - \nabla_x f(x^k; \theta^*))] \\ &\leq a_k - a_{k+1} + \frac{1}{2} \gamma_x^2 M^2 + \frac{1}{2} \gamma_x^{2-\tau} \mathbb{E}[\|x^k - x^*\|^2] \\ &\quad + \frac{1}{2} \gamma_x^\tau \mathbb{E}[\|\nabla_x f(x^k; \theta^k) - \nabla_x f(x^k; \theta^*)\|^2] \\ &\leq a_k - a_{k+1} + \frac{1}{2} \gamma_x^2 M^2 + \frac{1}{2} \gamma_x^{2-\tau} M_x^2 + \frac{1}{2} \gamma_x^\tau L_\theta^2 \mathbb{E}[\|\theta^k - \theta^*\|^2], \end{aligned}$$

where  $0 < \tau < 1$ . It follows that

$$\begin{aligned} \gamma_x \limsup_{k \rightarrow \infty} \mathbb{E}[f(x^k; \theta^*) - f(x^*; \theta^*)] &\leq \limsup_{k \rightarrow \infty} a_k - \limsup_{k \rightarrow \infty} a_{k+1} + \frac{1}{2} \gamma_x^2 M^2 + \frac{1}{2} \gamma_x^{2-\tau} M_x^2 \\ &\quad + \frac{1}{2} \gamma_x^\tau L_\theta^2 \limsup_{k \rightarrow \infty} \mathbb{E}[\|\theta^k - \theta^*\|^2] \\ &\leq \frac{1}{2} \gamma_x^2 M^2 + \frac{1}{2} \gamma_x^{2-\tau} M_x^2 + \frac{1}{2} \gamma_x^\tau L_\theta^2 \frac{\gamma_\theta \nu_\theta^2}{2\mu_\theta - \gamma_\theta C_\theta^2}. \end{aligned}$$

It follows that

$$\limsup_{k \rightarrow \infty} \mathbb{E}[f(x^k; \theta^*) - f(x^*; \theta^*)] \leq \frac{1}{2} \gamma_x M^2 + \frac{1}{2} \gamma_x^{1-\tau} M_x^2 + \frac{1}{2} \gamma_x^{\tau-1} L_\theta^2 \frac{\gamma_\theta \nu_\theta^2}{2\mu_\theta - \gamma_\theta C_\theta^2}.$$

By the Lipschitz continuity of  $f(x; \theta)$  in  $\theta$  (A6(i)), Hölder's inequality and (2.29), we have

$$\begin{aligned} \limsup_{k \rightarrow \infty} |\mathbb{E}[f(x^k; \theta^k) - f(x^k; \theta^*)]| &\leq D_\theta \limsup_{k \rightarrow \infty} \mathbb{E}[\|\theta^k - \theta^*\|] \\ &\leq D_\theta \limsup_{k \rightarrow \infty} \sqrt{\mathbb{E}[\|\theta^k - \theta^*\|^2]} \\ &= D_\theta \sqrt{\limsup_{k \rightarrow \infty} \mathbb{E}[\|\theta^k - \theta^*\|^2]} \\ &\leq D_\theta \sqrt{\frac{\gamma_\theta \nu_\theta^2}{2\mu_\theta - \gamma_\theta C_\theta^2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \limsup_{k \rightarrow \infty} |\mathbb{E}[f(x^k; \theta^k) - f(x^*; \theta^*)]| &\leq \limsup_{k \rightarrow \infty} |\mathbb{E}[f(x^k; \theta^k) - f(x^k; \theta^*)]| + \limsup_{k \rightarrow \infty} |\mathbb{E}[f(x^k; \theta^*) - f(x^*; \theta^*)]| \\ &\leq \frac{1}{2} \gamma_x M^2 + \frac{1}{2} \gamma_x^{1-\tau} M_x^2 + \frac{1}{2} \gamma_x^{\tau-1} L_\theta^2 \frac{\gamma_\theta \nu_\theta^2}{2\mu_\theta - \gamma_\theta C_\theta^2} + D_\theta \sqrt{\frac{\gamma_\theta \nu_\theta^2}{2\mu_\theta - \gamma_\theta C_\theta^2}}. \end{aligned}$$

□

**2.4. Regret analysis.** In this subsection, we consider the problem of online convex programming in a misspecified regime. In online convex programming problems, a decision-maker sees an infinite sequence

of functions  $c_1, c_2, \dots$  where each function is convex in its argument over a closed and convex set  $X$ . An *online* convex programming algorithm [50] generates an iterate  $x_k$  at each time epoch  $k$  and a metric of performance is the *regret* associated with not using an offline algorithm that considers the following problem:  $\min_{x \in X} \sum_{k=1}^K c_k(x)$ . If an online convex algorithm generates iterates  $x_1, x_2, \dots$ , then the regret  $R_K$  is defined as

$$R_K \triangleq \left[ \sum_{k=1}^K c_k(x_k) - \min_{x \in X} \sum_{k=1}^K c_k(x) \right].$$

A desirable feature of an online convex programming algorithm is that it is characterized by sublinear regret [50].

Often the model prescribed in an online optimization regime can be refined to a setting where the functions are related across time rather than being a sequence of unrelated functions. We consider one particular regime in which the decision-maker sees a sequence of functions given by  $f(\bullet; \theta_1), f(\bullet; \theta_2), \dots$ . Furthermore, neither the values  $\theta_1, \theta_2, \dots$  are known to the decision-maker nor is the fact that  $\theta_k \rightarrow \theta^*$  as  $k \rightarrow \infty$ . As earlier, we assume that the decision-maker has to furnish  $x_1, x_2, \dots$  and we define the *misspecified regret* after  $K$  steps associated with our generated sequence  $\{x^k, \theta^k\}$  as follows:

$$R_K \triangleq \mathbb{E} \left[ \sum_{k=1}^K f(x^k; \theta^k, \xi) - K f(x^*; \theta^*, \xi) \right].$$

Unlike the traditional definition, we consider the departure from  $f(x^*, \theta^*)$  and should be contrasted with the standard regret metric given by  $R_K^{\text{std}} \triangleq \mathbb{E} \left[ \sum_{k=1}^K f(x^k; \theta^*, \xi) - K f(x^*; \theta^*, \xi) \right]$ . For purposes of deriving analytical bounds, we define the following variant of regret as follows:

$$\widehat{R}_K \triangleq \mathbb{E} \left[ \sum_{k=1}^K f(x^k; \theta^k, \xi) - \sum_{k=1}^K f(y_K^*; \theta^k, \xi) \right], \text{ where } y_K^* \triangleq \underset{y \in X}{\operatorname{argmin}} \mathbb{E} \left[ \sum_{k=1}^K f(y; \theta^k, \xi) \right].$$

Next, we provide a rate of decay of the upper bound of average regret.

**THEOREM 3 (Regret under convexity of  $f$ ).** *Suppose (A3) and (A6) hold. Suppose  $\mathbb{E}[\|x - x^*\|^2] \leq M_x^2$ ,  $\mathbb{E}[\|\nabla_x f(x; \theta) + w^k\|^2] \leq M^2$  and  $\mathbb{E}[\|\nabla_\theta g(\theta) + v^k\|^2] \leq M_\theta^2$  for all  $x \in X$  and  $\theta \in \Theta$ . Suppose  $\mathbb{E}[\|\nabla_x f(y_K^*; \theta^k) + u^k\|^2] \leq M^2$ , where  $u^k \triangleq \mathbb{E}[\nabla_x f(y_K^*; \theta^k, \xi)] - \nabla_x f(y_K^*; \theta^k)$ . Let  $\{x^k, \theta^k\}$  be computed via Algorithm 1. Suppose  $\gamma_{k,x} = k^{-\alpha}$  with  $0.5 \leq \alpha < 1$ , and  $\gamma_{\theta,k} = \lambda_\theta/k$  with  $\lambda_\theta > 1/(2\mu_\theta)$ . If  $0 < \beta < 1$ , then the following holds:*

$$\frac{R_K}{K} \leq \frac{M_x^2 K^{\alpha-1}}{2} + \frac{M^2(K^{1-\alpha} - \alpha)}{2(1-\alpha)K} + \frac{D_\theta \sqrt{Q_\theta(\lambda_\theta)}(2\sqrt{K} - 1)}{K} + \frac{M_x^2}{2K^\beta} + \frac{L_\theta^2 Q_\theta(\lambda_\theta)(\ln(K) + 1)}{2K^{1-\beta}},$$

where  $\beta > 0$ . Furthermore,

$$\limsup_{K \rightarrow \infty} \frac{R(K)}{K} \leq 0.$$

*Proof.* By using the proof in Theorem 1 in [50] (cf. Theorem 7 in Appendix A), we obtain that  $\widehat{R}_K/K$

is bounded as follows:

$$\widehat{R}_K \leq \frac{M_x^2}{2\gamma_{K,x}} + \frac{M^2}{2} \sum_{k=1}^K \gamma_{k,x}.$$

Next, if  $\gamma_{k,x} = k^{-\alpha}$  with  $0.5 \leq \alpha < 1$ , then we have the following bound on  $\sum_{k=1}^K \gamma_{k,x}$ :

$$\sum_{k=1}^K \gamma_{k,x} = \sum_{k=1}^K k^{-\alpha} \leq 1 + \int_1^K x^{-\alpha} dx = \frac{1}{1-\alpha} (K^{1-\alpha} - \alpha).$$

Therefore, we obtain the following bound on  $\widehat{R}_K$ :

$$(2.30) \quad \widehat{R}_K \leq \frac{M_x^2 K^\alpha}{2} + \frac{M^2 (K^{1-\alpha} - \alpha)}{2(1-\alpha)}.$$

Recall that the difference between the real regret and misspecified regret is given by the following:

$$\begin{aligned} |R_K - \widehat{R}_K| &= \left| \mathbb{E} \left[ \sum_{k=1}^K f(y_K^*; \theta^k, \xi) - K f(x^*; \theta^*, \xi) \right] \right| \\ &\leq \left| \mathbb{E} \left[ \sum_{k=1}^K f(y_K^*; \theta^k, \xi) - K f(y_K^*; \theta^*, \xi) \right] \right| + |\mathbb{E} [K (f(y_K^*; \theta^*, \xi) - f(x^*; \theta^*, \xi))]|, \end{aligned}$$

or

$$(2.31) \quad \frac{|R_K - \widehat{R}_K|}{K} \leq \underbrace{\left| \mathbb{E} \left[ \frac{1}{K} \sum_{k=1}^K f(y_K^*; \theta^k, \xi) - f(y_K^*; \theta^*, \xi) \right] \right|}_{\text{Term 1}} + \underbrace{|\mathbb{E} [f(y_K^*; \theta^*, \xi) - f(x^*; \theta^*, \xi)]|}_{\text{Term 2}}.$$

We proceed to derive bounds for **Terms 1** and **2**. **Term 1** in (2.31) may be bounded as follows:

$$\begin{aligned} \left| \mathbb{E} \left[ \frac{1}{K} \sum_{k=1}^K f(y_K^*; \theta^k, \xi) - f(y_K^*; \theta^*, \xi) \right] \right| &\leq \frac{1}{K} \sum_{k=1}^K \mathbb{E} [|f(y_K^*; \theta^k, \xi) - f(y_K^*; \theta^*, \xi)|] \\ &\leq \frac{D_\theta}{K} \sum_{k=1}^K \mathbb{E} [\|\theta^k - \theta^*\|] \\ &\leq \frac{D_\theta}{K} \sum_{k=1}^K \sqrt{\frac{Q_\theta(\lambda_\theta)}{k}}. \end{aligned}$$

where the second and third inequalities follow from the Lipschitz continuity of  $\nabla f(y^*; \theta)$  in  $\theta$  (A6) and (2.19).

Through some analysis, the right hand side may be further bounded as follows:

$$(2.32) \quad \frac{D_\theta}{K} \sum_{k=1}^K \sqrt{\frac{Q_\theta(\lambda_\theta)}{k}} \leq \frac{D_\theta \sqrt{Q_\theta(\lambda_\theta)}}{K} \left( 1 + \int_1^K \frac{1}{\sqrt{x}} dx \right) \leq \frac{D_\theta \sqrt{Q_\theta(\lambda_\theta)} (2\sqrt{K} - 1)}{K}.$$

This implies that **Term 1** in (2.31) converges to zero as  $K \rightarrow \infty$ . Next, we consider **Term 2** in (2.31). By the optimality condition for  $y_K^*$ , we have the following expression:

$$\begin{aligned} 0 &\geq \sum_{k=1}^K \mathbb{E} [(y_K^* - x^*)^T \nabla_x f(y_K^*; \theta^k, \xi)] \\ (2.33) \quad &= \sum_{k=1}^K \mathbb{E} [(y_K^* - x^*)^T \nabla_x f(y_K^*; \theta^*, \xi)] + \sum_{k=1}^K \mathbb{E} [(y_K^* - x^*)^T (\nabla_x f(y_K^*; \theta^k, \xi) - \nabla_x f(y_K^*; \theta^*, \xi))]. \end{aligned}$$

Since  $f(x; \theta)$  is convex in  $x$  for every  $\theta \in \Theta$ , we may leverage the gradient inequality.

$$(2.34) \quad \begin{aligned} & \mathbb{E}[f(x^*; \theta^*, \xi)] \geq \mathbb{E}[f(y_K^*; \theta^*, \xi)] + \mathbb{E}[\nabla_x f(y_K^*; \theta^*, \xi)^T (x^* - y_K^*)] \\ \implies & \mathbb{E}[(y_K^* - x^*)^T \nabla_x f(y_K^*; \theta^*, \xi)] \geq \mathbb{E}[f(y_K^*; \theta^*, \xi) - f(x^*; \theta^*, \xi)]. \end{aligned}$$

Combining (2.33) and (2.34), we get the following lower bound:

$$0 \geq \sum_{k=1}^K \mathbb{E}[f(y_K^*; \theta^*, \xi) - f(x^*; \theta^*, \xi)] + \sum_{k=1}^K \mathbb{E}[(y_K^* - x^*)^T (\nabla_x f(y_K^*; \theta^k, \xi) - \nabla_x f(y_K^*; \theta^*, \xi))].$$

This allows for constructing the following bound on  $\sum_{k=1}^K \mathbb{E}[f(y_K^*; \theta^*, \xi) - f(x^*; \theta^*, \xi)]$ :

$$(2.35) \quad \begin{aligned} \sum_{k=1}^K \mathbb{E}[f(y_K^*; \theta^*, \xi) - f(x^*; \theta^*, \xi)] & \leq - \sum_{k=1}^K \mathbb{E}[(y_K^* - x^*)^T (\nabla_x f(y_K^*; \theta^k, \xi) - \nabla_x f(y_K^*; \theta^*, \xi))] \\ & \leq \frac{1}{2} \sum_{k=1}^K \delta_K \mathbb{E}[\|y_K^* - x^*\|^2] + \frac{1}{2} \sum_{k=1}^K \frac{1}{\delta_K} \mathbb{E}[\|\nabla_x f(y_K^*; \theta^k, \xi) - \nabla_x f(y_K^*; \theta^*, \xi)\|^2] \\ & \leq \frac{1}{2} \sum_{k=1}^K \delta_K M_x^2 + \frac{1}{2} \sum_{k=1}^K \frac{1}{\delta_K} L_\theta^2 \mathbb{E}[\|\theta^k - \theta^*\|^2] \\ & \leq \frac{1}{2} \sum_{k=1}^K \delta_K M_x^2 + \frac{1}{2} \sum_{k=1}^K \frac{1}{\delta_K} \frac{L_\theta^2 Q_\theta(\lambda_\theta)}{k}, \end{aligned}$$

where  $\delta_K = K^{-\beta}$  with  $0 < \beta < 1$  and the last inequality follows from (2.19). Note that  $\sum_{k=1}^K \frac{1}{k} \leq \ln(K) + 1$ .

$$(2.36) \quad \begin{aligned} \text{Thus, } |\mathbb{E}[f(y_K^*; \theta^*, \xi) - f(x^*; \theta^*, \xi)]| & = \mathbb{E}[f(y_K^*; \theta^*) - f(x^*; \theta^*)] \\ & \leq \frac{M_x^2}{2K^\beta} + \frac{\sum_{k=1}^K \frac{L_\theta^2 Q_\theta(\lambda_\theta)}{k}}{2K\delta_K} \\ & \leq \frac{M_x^2}{2K^\beta} + \frac{L_\theta^2 Q_\theta(\lambda_\theta)(\ln(K) + 1)}{2K^{1-\beta}}. \end{aligned}$$

Combining (2.30), (2.31), (2.32), and (2.36), we have that  $R_K/K$  can be bounded as follows:

$$\begin{aligned} \frac{R_K}{K} & \leq \frac{\hat{R}_K}{K} + \frac{R_K - \hat{R}_K}{K} \leq \frac{\hat{R}_K}{K} + \frac{|R_K - \hat{R}_K|}{K} \\ & \leq \frac{M_x^2 K^{\alpha-1}}{2} + \frac{M^2(K^{1-\alpha} - \alpha)}{2(1-\alpha)K} + \frac{D_\theta \sqrt{Q_\theta(\lambda_\theta)}(2\sqrt{K} - 1)}{K} + \frac{M_x^2}{2K^\beta} + \frac{L_\theta^2 Q_\theta(\lambda_\theta)(\ln(K) + 1)}{2K^{1-\beta}}. \end{aligned}$$

Furthermore, this implies that the limit superior of the average regret is nonpositive.  $\square$

**Remark:** In effect, in the context of learning and optimization, the averaging approach leads to a complexity bound given loosely by

$$\mathcal{O} \left( \frac{a}{K^{1-\alpha}} + \frac{b}{K^\alpha} + \frac{d}{K^\beta} + \underbrace{\frac{c_\theta}{\sqrt{K}} + \frac{e_\theta \ln(K)}{K^{1-\beta}}}_{\text{contribution from learning}} \right),$$

where  $a, b, c_\theta, d, e_\theta$  are suitably defined. If  $\theta^*$  is available, then  $c_\theta, e_\theta = 0$ . Furthermore, by setting  $\alpha = 0.5$  and  $\beta = 0.5$ , this leads to the bound of  $\mathcal{O}(\ln K / \sqrt{K})$ , which is a degradation as the result of learning  $\theta^*$ .  $\blacksquare$

**3. Stochastic variational inequality problems with imperfect information.** Several shortcomings exist in the optimization based formulation represented by  $(\mathcal{P}_x^o(\theta^*))$ . First, the misspecification arises entirely in the objectives while the constraints are known with certainty. Second, the underlying problem need not be an optimization problem, but could instead be captured by a variational inequality problem. Such problems [15] can capture a range of problems including economic equilibrium problems, traffic equilibrium problems, and convex Nash games. In fact, variational inequality problems can effectively capture optimization problems with misspecified constraints. This motivates the consideration of the misspecified stochastic variational inequality problem  $(\mathcal{P}_x^v(\theta^*))$  where  $\theta^*$  can be learnt through the solution of the following problem:

$$(\mathcal{L}_\theta^v) \quad (\vartheta - \theta)^T \mathbb{E}[G(\theta; \eta)] \geq 0, \quad \forall \vartheta \in \Theta,$$

where  $G : \theta \times \mathbb{R}^p \rightarrow \mathbb{R}^m$ , and  $\Theta$  and  $\eta$  abide by the previous specifications. In the majority of problem settings,  $G(\theta; \theta) \triangleq \nabla_\theta g(\theta; \eta)$  but we employ the variational structure to introduce generality. In this section, we extend the results of the previous section to this regime. Specifically, we develop the convergence theory under settings where the variational map  $F$  is both strongly monotone and merely monotone in  $x$  for every  $\theta \in \Theta$  in Section 3.1 and provide rate statements in Section 3.2.

**3.1. Almost-sure convergence.** As in Section 2, we propose a set of coupled stochastic approximation schemes for computing  $x^*$  and  $\theta^*$ . Given  $x^0 \in X$  and  $\theta^0 \in \Theta$ , the coupled SA schemes are stated next:

**ALGORITHM 2 (Coupled SA schemes for stochastic variational inequality problems).**

**Step 0.** Given  $x_0 \in X, \theta_0 \in \Theta$  and sequences  $\{\gamma_{k,x}, \gamma_{k,\theta}\}$ ,  $k := 0$

**Step 1.**

$$(\text{Comp}_k) \quad x^{k+1} := \Pi_X (x^k - \gamma_{k,x}(F(x^k; \theta^k) + w^k))$$

$$(\text{Learn}_k) \quad \theta^{k+1} := \Pi_\Theta (\theta^k - \gamma_{k,\theta}(G(\theta^k) + v^k)),$$

where  $w^k \triangleq F(x^k; \theta^k, \xi^k) - F(x^k; \theta^k)$  and  $v^k \triangleq G(\theta^k; \eta^k) - G(\theta^k)$ .

**Step 2.** If  $k > K$ , stop; else  $k : k + 1$ , go to Step. 1.

We begin by stating an assumption similar to (A1-1) on the mappings  $F$  and  $G$ .

**ASSUMPTION 7 (A1-3).** Suppose the following hold:

- (i) For every  $\theta \in \Theta$ ,  $F(x; \theta)$  is both strongly monotone and Lipschitz continuous in  $x$  with constants  $\mu_x$  and  $L_x$ , respectively.
- (ii) For every  $x \in X$ ,  $F(x; \theta)$  is Lipschitz continuous in  $\theta$  with constant  $L_\theta$ .
- (iii)  $G(\theta)$  is strongly monotone and Lipschitz continuous in  $\theta$  with constants  $\mu_\theta$  and  $C_\theta$ , respectively.

Now, we can leverage the results in Section 2.2 to examine the convergence properties for Algorithm 2.

**PROPOSITION 4 (Almost-sure convergence under strong monotonicity of  $F$ ).** Suppose (A1-3), (A2-1) and (A3) hold. Let  $\{x^k, \theta^k\}$  be computed via Algorithm 2. Then,  $x^k \rightarrow x^*$  a.s. and  $\theta^k \rightarrow \theta^*$  a.s. as  $k \rightarrow \infty$ , where  $x^*$  is the unique solution to  $(\mathcal{P}_x^v(\theta^*))$  and  $\theta^*$  is the unique solution to  $(\mathcal{L}_\theta^v)$ .

*Proof.* Note that  $x^* = \Pi_X(x^* - \gamma_{k,x}F(x^*; \theta^*))$  and  $\theta^* = \Pi_\Theta(\theta^* - \gamma_{k,\theta}G(\theta^*))$ . If we replace  $\nabla_x f$  and  $\nabla_\theta g$  by  $F$  and  $G$  in Proposition 1, respectively, then by the proof of Proposition 1, we get  $x^k \rightarrow x^*$  a.s. and  $\theta^k \rightarrow \theta^*$  a.s. as  $k \rightarrow \infty$ .  $\square$

Next, we weaken the rather stringent requirement of strong monotonicity of the map by using an iterative Tikhonov regularization, which can be stated as follows.

**ALGORITHM 3 (Coupled regularized SA schemes for stochastic variational inequality problems).**

**Step 0.** Given  $x_0 \in X, \theta_0 \in \Theta$  and sequences  $\{\gamma_{k,x}, \gamma_{k,\theta}\}$ ,  $k := 0$

**Step 1.**

$$(\text{Comp}_k) \quad x^{k+1} := \Pi_X(x^k - \gamma_{k,x}(F(x^k; \theta^k) + \epsilon_k x^k + w^k))$$

$$(\text{Learn}_k) \quad \theta^{k+1} := \Pi_\Theta(\theta^k - \gamma_{k,\theta}(G(\theta^k) + v^k)),$$

where  $w^k \triangleq F(x^k; \theta^k, \xi^k) - F(x^k; \theta^k)$  and  $v^k \triangleq G(\theta^k; \eta^k) - G(\theta^k)$ .

**Step 2.** If  $k > K$ , stop; else  $k := k + 1$ , go to Step. 1.

Unlike in standard Tikhonov regularization, such a scheme updates the regularization parameter  $\epsilon_k$  after every step. Tikhonov regularization and its iterative counterpart has a long history [41] while iterative regularization schemes have seen relatively less study in the context of variational inequality problems (cf. [31, 29]). Of note is the extension to distributed schemes to accommodate monotone Cartesian stochastic variational inequality problems [32]. We employ such techniques in developing single-loop stochastic approximation schemes in the context of learning and optimization. The following assumptions will be made on both the decision variable and parameter.

**ASSUMPTION 8 (A1-4).** Suppose the following holds in addition to (A1-3 (ii)) and (A1-3 (iii)).

(i) For every  $\theta \in \Theta$ ,  $F(x; \theta)$  is monotone in  $x$  and Lipschitz continuous in  $x$  with constant  $L_x$ .

In iterative Tikhonov regularization, one cannot independently choose  $\{\epsilon_k\}$  and  $\{\gamma_k\}$ ; in fact, these sequences are related and satisfy some collectively imposed requirements.

**ASSUMPTION 9 (A2-3).** Let  $\{\gamma_{k,x}\}$ ,  $\{\gamma_{k,\theta}\}$ ,  $\{\epsilon_k\}$  and some constant  $\tau \in (0, 1)$  be chosen such that:

- (i)  $\sum_{k=0}^{\infty} \gamma_{k,x}^{2-\tau} < \infty$  and  $\sum_{k=0}^{\infty} \gamma_{k,\theta}^2 < \infty$ ,
- (ii)  $\sum_{k=0}^{\infty} \gamma_{k,x} \epsilon_k = \infty$  and  $\sum_{k=0}^{\infty} \gamma_{k,\theta} = \infty$ ,
- (iii)  $\beta_k = \frac{\gamma_{k,x}}{2\gamma_{k,\theta}\mu_\theta} \downarrow 0$  as  $k \rightarrow \infty$ .
- (iv)  $\sum_{k=0}^{\infty} \frac{(\epsilon_{k-1} - \epsilon_k)}{\epsilon_k} < \infty$ .

Before providing a convergence result for Algorithm 3, we introduce the following results.

**LEMMA 3.1.** Let  $H : K \rightarrow \mathbb{R}^n$  be a mapping that is monotone over  $K$ , and Lipschitz continuous over  $K$  with constant  $L$ . Then, for any  $\gamma > 0$  and  $\epsilon > 0$ , we have  $\|(x - y) - \gamma(H(x) - H(y)) - \epsilon\gamma(x - y)\| \leq q\|x - y\|$ , where  $q = \sqrt{1 - 2\gamma\epsilon + \gamma^2(L^2 + \epsilon^2)}$ .

*Proof.* See proof of Theorem 1 in [28].  $\square$

**LEMMA 3.2.** Let  $H : K \rightarrow \mathbb{R}^n$  be a mapping that is monotone over  $K$ . Given  $\epsilon_k > 0$ , let  $y^k$  be a solution to  $VI(K, H + \epsilon_k \mathbf{I})$ . Then,

$$\|y^k - y^{k-1}\| \leq \frac{M(\epsilon_{k-1} - \epsilon_k)}{\epsilon_k},$$

where  $M = \|x^*\|$  and  $x^*$  is a solution to  $VI(H, K)$ .

*Proof.* See Lemma 3 in [28].  $\square$

The convergence result for Algorithm 3 can be stated as follows.

**THEOREM 4 (Almost-sure convergence under monotonicity of  $F$ ).** Suppose (A1-4), (A2-3) and (A3) hold. Suppose  $X$  is bounded and the solution set  $X^*$  of  $(\mathcal{P}_x^v(\theta^*))$  is nonempty. Let  $\{x^k, \theta^k\}$  be computed via Algorithm 3. Then,  $\theta^k \rightarrow \theta^*$  a.s. as  $k \rightarrow \infty$ , and  $x^k$  converges to a random point in  $X^*$  a.s. as  $k \rightarrow \infty$ .

*Proof.* We have for any  $x^* \in X^*$  that  $x^* = \Pi_X(x^* - \gamma_{k,x}F(x^*; \theta^*))$ . Suppose  $y^k$  is a solution to the following fixed-point problem

$$y^k = \Pi_X(y^k - \gamma_{k,x}(F(y^k; \theta^*) + \epsilon_k y^k)).$$

Then, by the triangle inequality  $\|x^{k+1} - x^*\|$  may be bounded as follows:

$$\|x^{k+1} - x^*\| \leq \underbrace{\|x^{k+1} - y^k\|}_{\text{Term 1}} + \underbrace{\|y^k - x^*\|}_{\text{Term 2}}.$$

Term 2 converges to zero by the convergence statement of Tikhonov regularization methods [15]. By using the non-expansivity of the Euclidean projector,  $\|x^{k+1} - y^k\|^2$  can be bounded as follows:

$$\begin{aligned} \|x^{k+1} - y^k\|^2 &= \|\Pi_X(x^k - \gamma_{k,x}(F(x^k; \theta^k) + \epsilon_k x^k + w^k)) - \Pi_X(y^k - \gamma_{k,x}(F(y^k; \theta^*) + \epsilon_k y^k))\|^2 \\ &\leq \|(x^k - y^k) - \gamma_{k,x}(F(x^k; \theta^k) - F(y^k; \theta^*)) - \epsilon_k \gamma_{k,x}(x^k - y^k) - \gamma_{k,x}w^k\|^2. \end{aligned}$$

By adding and subtracting  $\gamma_{k,x}F(x^k; \theta^*)$ , this expression can be further expanded as follows:

$$\begin{aligned} &\|(x^k - y^k) - \gamma_{k,x}(F(x^k; \theta^k) - F(y^k; \theta^*)) - \gamma_{k,x}(F(x^k; \theta^k) - F(x^k; \theta^*)) - \epsilon_k \gamma_{k,x}(x^k - y^k) - \gamma_{k,x}w^k\|^2 \\ &= \|(x^k - y^k) - \gamma_{k,x}(F(x^k; \theta^k) - F(y^k; \theta^*)) - \epsilon_k \gamma_{k,x}(x^k - y^k)\|^2 + \gamma_{k,x}^2 \|F(x^k; \theta^k) - F(x^k; \theta^*)\|^2 + \gamma_{k,x}^2 \|w^k\|^2 \\ &\quad - 2[(x^k - y^k) - \gamma_{k,x}(F(x^k; \theta^k) - F(y^k; \theta^*)) - \epsilon_k \gamma_{k,x}(x^k - y^k)]^T \times (F(x^k; \theta^k) - F(x^k; \theta^*)) \\ &\quad - 2[(x^k - y^k) - \gamma_{k,x}(F(x^k; \theta^k) - F(y^k; \theta^*)) - \epsilon_k \gamma_{k,x}(x^k - y^k)]^T w^k + 2\gamma_{k,x}^2 (F(x^k; \theta^k) - F(x^k; \theta^*))^T w^k. \end{aligned}$$

Noting that  $\mathbb{E}[w^k | \mathcal{F}_k] = 0$ , we have

$$(3.1) \quad \mathbb{E}[\|x^{k+1} - y^k\|^2 | \mathcal{F}_k] \leq \text{Term 3} + \text{Term 4} + \text{Term 5} + \gamma_{k,x}^2 \mathbb{E}[\|w^k\|^2 | \mathcal{F}_k],$$

where

$$\text{Term 3} \triangleq \|(x^k - y^k) - \gamma_{k,x}(F(x^k; \theta^k) - F(y^k; \theta^*)) - \epsilon_k \gamma_{k,x}(x^k - y^k)\|^2,$$

$$\text{Term 4} \triangleq \gamma_{k,x}^2 \|F(x^k; \theta^k) - F(x^k; \theta^*)\|^2,$$

$$\text{Term 5} \triangleq -2\gamma_{k,x}[(x^k - y^k) - \gamma_{k,x}(F(x^k; \theta^k) - F(y^k; \theta^*)) - \epsilon_k \gamma_{k,x}(x^k - y^k)]^T (F(x^k; \theta^k) - F(x^k; \theta^*)).$$

By Lemma 3.1 and (A1-4), Term 3 can be further bounded by

$$(3.2) \quad (1 - 2\gamma_{k,x}\epsilon_k + \gamma_{k,x}^2(L_x^2 + (\epsilon_k)^2))\|x^k - y^k\|^2.$$

By the Lipschitz continuity of  $F(x; \theta)$  in  $\theta$  (A1-4), Term 4 can be further bounded by

$$(3.3) \quad \gamma_{k,x}^2 L_\theta^2 \|\theta^k - \theta^*\|^2.$$

By the Cauchy-Schwarz inequality, Lemma 3.1, (A1-4) as well as the fact that  $2ab \leq a^2 + b^2$ , Term 5 can be further bounded by

$$\begin{aligned}
(3.4) \quad & 2\gamma_{k,x} \|(x^k - y^k) - \gamma_{k,x}(F(x^k; \theta^*) - F(y^k; \theta^*)) - \epsilon_k \gamma_{k,x}(x^k - y^k)\| \|F(x^k; \theta^k) - F(x^k; \theta^*)\| \\
& \leq 2\gamma_{k,x} \sqrt{1 - 2\gamma_{k,x}\epsilon_k + \gamma_{k,x}^2(L_x^2 + (\epsilon_k)^2)} \|x^k - y^k\| L_\theta \|\theta^k - \theta^*\| \\
& \leq 2\gamma_{k,x} L_\theta \|x^k - y^k\| \|\theta^k - \theta^*\| \\
& \leq \gamma_{k,x}^{2-\tau} L_\theta^2 \|x^k - y^k\|^2 + \gamma_{k,x}^\tau \|\theta^k - \theta^*\|^2,
\end{aligned}$$

where  $\tau \in (0, 1)$  is chosen to satisfy (A2-3). Combining (3.1), (3.2), (3.3) and (3.4), we get

$$(3.5) \quad \mathbb{E}[\|x^{k+1} - y^k\|^2 \mid \mathcal{F}_k] \leq (q_{k,x}^2 + \gamma_{k,x}^{2-\tau} L_\theta^2) \|x^k - y^k\|^2 + (\gamma_{k,x}^\tau + \gamma_{k,x}^2 L_\theta^2) \|\theta^k - \theta^*\|^2 + \gamma_{k,x}^2 \nu_x^2,$$

where  $q_{k,x} = \sqrt{1 - 2\gamma_{k,x}\epsilon_k + \gamma_{k,x}^2(L_x^2 + (\epsilon_k)^2)}$ .

On the other hand, we have that  $\theta^*$  is the unique solution to  $\text{VI}(\Theta, \mathbb{E}[G(\bullet; \eta)])$  and

$$\theta^* = \Pi_\Theta(\theta^* - \gamma_{k,\theta} G(\theta^*)).$$

Therefore, by the nonexpansivity of the Euclidean projector,  $\|\theta^{k+1} - \theta^*\|^2$  may be bounded as follows:

$$\begin{aligned}
& \|\theta^{k+1} - \theta^*\|^2 = \|\Pi_\Theta(\theta^k - \gamma_{k,\theta}(G(\theta^k) + v^k)) - \Pi_\Theta(\theta^* - \gamma_{k,\theta} G(\theta^*))\|^2 \\
& \leq \|(\theta^k - \theta^*) - \gamma_{k,\theta}(G(\theta^k) - G(\theta^*)) - \gamma_{k,\theta} v^k\|^2 \\
& = \|(\theta^k - \theta^*) - \gamma_{k,\theta}(G(\theta^k) - G(\theta^*))\|^2 + \gamma_{k,\theta}^2 \|v^k\|^2 - 2\gamma_{k,\theta} [(\theta^k - \theta^*) - \gamma_{k,\theta}(G(\theta^k) - G(\theta^*))]^T v^k.
\end{aligned}$$

By taking conditional expectations and by recalling that  $\mathbb{E}[v^k \mid \mathcal{F}_k] = 0$  (A3), we obtain the following:

$$\begin{aligned}
(3.6) \quad & \mathbb{E}[\|\theta^{k+1} - \theta^*\|^2 \mid \mathcal{F}_k] \leq \|(\theta^k - \theta^*) - \gamma_{k,\theta}(G(\theta^k) - G(\theta^*))\|^2 + \gamma_{k,\theta}^2 \mathbb{E}[\|v^k\|^2 \mid \mathcal{F}_k] \\
& \leq q_{k,\theta}^2 \|\theta^k - \theta^*\|^2 + \gamma_{k,\theta}^2 \nu_\theta^2,
\end{aligned}$$

where  $q_{k,\theta} = \sqrt{1 - 2\gamma_{k,\theta}\mu_\theta + \gamma_{k,\theta}^2 C_\theta^2}$ , and the second inequality follows from Lemma 2.4, (A1-4) and (A3). Since by (A2-3)  $\sum_{k=0}^\infty (1 - q_{k,\theta}^2) = \infty$  and  $\sum_{k=0}^\infty \gamma_{k,\theta}^2 \nu_\theta^2 < \infty$ , and

$$\lim_{k \rightarrow \infty} \frac{\gamma_{k,\theta}^2 \nu_\theta^2}{1 - q_{k,\theta}^2} = \lim_{k \rightarrow \infty} \frac{\gamma_{k,\theta}^2 \nu_\theta^2}{2\gamma_{k,\theta}\mu_\theta - \gamma_{k,\theta}^2 C_\theta^2} = \lim_{k \rightarrow \infty} \frac{\gamma_{k,\theta} \nu_\theta^2}{2\mu_\theta - \gamma_{k,\theta} C_\theta^2} = 0,$$

we have by Lemma 2.2 that  $\|\theta^k - \theta^*\| \rightarrow 0$  a.s. as  $k \rightarrow \infty$ . Choose  $\beta_k = \frac{\gamma_{k,x}^\tau}{2\gamma_{k,\theta}\mu_\theta}$  by (A2-3). Note that by assumption  $\beta_{k+1} \leq \beta_k$ . By multiplying the left hand side of (3.6) by  $\beta_{k+1}$  and adding to the left hand side of (3.5), we get

$$\begin{aligned}
(3.7) \quad & \mathbb{E}[\|x^{k+1} - y^k\|^2 \mid \mathcal{F}_k] + \beta_{k+1} \mathbb{E}[\|\theta^{k+1} - \theta^*\|^2 \mid \mathcal{F}_k] \\
& \leq \mathbb{E}[\|x^{k+1} - y^k\|^2 \mid \mathcal{F}_k] + \beta_k \mathbb{E}[\|\theta^{k+1} - \theta^*\|^2 \mid \mathcal{F}_k] \\
& \leq (q_{k,x}^2 + \gamma_{k,x}^{2-\tau} L_\theta^2) \|x^k - y^k\|^2 + (\beta_k q_{k,\theta}^2 + \gamma_{k,x}^\tau + \gamma_{k,x}^2 L_\theta^2) \|\theta^k - \theta^*\|^2 + \beta_k \gamma_{k,\theta}^2 \nu_\theta^2 + \gamma_{k,x}^2 \nu_x^2 \\
& = (q_{k,x}^2 + \gamma_{k,x}^{2-\tau} L_\theta^2) \|x^k - y^k\|^2 + \underbrace{\frac{\beta_k q_{k,\theta}^2 + \gamma_{k,x}^\tau + \gamma_{k,x}^2 L_\theta^2}{\beta_k}}_{\text{Term6}} \beta_k \|\theta^k - \theta^*\|^2 + \beta_k \gamma_{k,\theta}^2 \nu_\theta^2 + \gamma_{k,x}^2 \nu_x^2.
\end{aligned}$$



Term 6 on the right hand side of (3.7) can be further expanded as

$$(3.8) \quad \begin{aligned} \frac{\beta_k q_{k,\theta}^2 + \gamma_{k,x}^\tau + \gamma_{k,x}^2 L_\theta^2}{\beta_k} &= q_{k,\theta}^2 + \frac{\gamma_{k,x}^\tau + \gamma_{k,x}^2 L_\theta^2}{\beta_k} = 1 - 2\gamma_{k,\theta}\mu_\theta + \gamma_{k,\theta}^2 C_\theta^2 + \frac{\gamma_{k,x}^\tau}{\beta_k} + \frac{\gamma_{k,x}^2 L_\theta^2}{\beta_k} \\ &= 1 + \gamma_{k,\theta}^2 C_\theta^2 + 2\gamma_{k,\theta}\gamma_{k,x}^{2-\tau}\mu_\theta L_\theta^2. \end{aligned}$$

Combining (3.7) and (3.8), we get

$$\begin{aligned} &\mathbb{E}[\|x^{k+1} - y^k\|^2 \mid \mathcal{F}_k] + \beta_{k+1}\mathbb{E}[\|\theta^{k+1} - \theta^*\|^2 \mid \mathcal{F}_k] \\ &\leq (q_{k,x}^2 + \gamma_{k,x}^{2-\tau} L_\theta^2)\|x^k - y^k\|^2 + (1 + \gamma_{k,\theta}^2 C_\theta^2 + 2\gamma_{k,\theta}\gamma_{k,x}^{2-\tau}\mu_\theta L_\theta^2)\beta_k\|\theta^k - \theta^*\|^2 + \beta_k\gamma_{k,\theta}^2\nu_\theta^2 + \gamma_{k,x}^2\nu_x^2 \\ &= (1 + \gamma_{k,\theta}^2 C_\theta^2 + 2\gamma_{k,\theta}\gamma_{k,x}^{2-\tau}\mu_\theta L_\theta^2)(\|x^k - y^k\|^2 + \beta_k\|\theta^k - \theta^*\|^2) - (\gamma_{k,\theta}^2 C_\theta^2 + 2\gamma_{k,\theta}\gamma_{k,x}^{2-\tau}\mu_\theta L_\theta^2 + 2\gamma_{k,x}\epsilon_k)\|x^k - y^k\|^2 \\ &\quad + (\gamma_{k,x}^2(L_x^2 + (\epsilon_k)^2) + \gamma_{k,x}^{2-\tau} L_\theta^2)\|x^k - y^k\|^2 + \beta_k\gamma_{k,\theta}^2\nu_\theta^2 + \gamma_{k,x}^2\nu_x^2. \end{aligned}$$

Note that  $\|x^{k+1} - y^k\|^2 \leq \|x^k - y^{k-1}\|^2 + 2\|x^k - y^{k-1}\|\|y^k - y^{k-1}\| + \|y^k - y^{k-1}\|^2$ . We have

$$\begin{aligned} &\mathbb{E}[\|x^{k+1} - y^k\|^2 \mid \mathcal{F}_k] + \beta_{k+1}\mathbb{E}[\|\theta^{k+1} - \theta^*\|^2 \mid \mathcal{F}_k] \\ &\leq (1 + \gamma_{k,\theta}^2 C_\theta^2 + 2\gamma_{k,\theta}\gamma_{k,x}^{2-\tau}\mu_\theta L_\theta^2)(\|x^k - y^{k-1}\|^2 + \beta_k\|\theta^k - \theta^*\|^2) \\ &\quad + 2(1 + \gamma_{k,\theta}^2 C_\theta^2 + 2\gamma_{k,\theta}\gamma_{k,x}^{2-\tau}\mu_\theta L_\theta^2)\|x^k - y^{k-1}\|\|y^k - y^{k-1}\| + (1 + \gamma_{k,\theta}^2 C_\theta^2 + 2\gamma_{k,\theta}\gamma_{k,x}^{2-\tau}\mu_\theta L_\theta^2)\|y^k - y^{k-1}\|^2 \\ &\quad - (\gamma_{k,\theta}^2 C_\theta^2 + 2\gamma_{k,\theta}\gamma_{k,x}^{2-\tau}\mu_\theta L_\theta^2 + 2\gamma_{k,x}\epsilon_k)\|x^k - y^k\|^2 + (\gamma_{k,x}^2(L_x^2 + (\epsilon_k)^2) + \gamma_{k,x}^{2-\tau} L_\theta^2)\|x^k - y^k\|^2 + \beta_k\gamma_{k,\theta}^2\nu_\theta^2 + \gamma_{k,x}^2\nu_x^2, \end{aligned}$$

which can be further reduced to

$$\begin{aligned} &\mathbb{E}[\|x^{k+1} - y^k\|^2 \mid \mathcal{F}_k] + \beta_{k+1}\mathbb{E}[\|\theta^{k+1} - \theta^*\|^2 \mid \mathcal{F}_k] \\ &\leq (1 + \gamma_{k,\theta}^2 C_\theta^2 + 2\gamma_{k,\theta}\gamma_{k,x}^{2-\tau}\mu_\theta L_\theta^2)(\|x^k - y^{k-1}\|^2 + \beta_k\|\theta^k - \theta^*\|^2) \\ &\quad + 2(1 + \gamma_{k,\theta}^2 C_\theta^2 + 2\gamma_{k,\theta}\gamma_{k,x}^{2-\tau}\mu_\theta L_\theta^2)\|x^k - y^{k-1}\|\|y^k - y^{k-1}\| \\ &\quad + (1 + \gamma_{k,\theta}^2 C_\theta^2 + 2\gamma_{k,\theta}\gamma_{k,x}^{2-\tau}\mu_\theta L_\theta^2)\|y^k - y^{k-1}\|^2 \\ &\quad - 2\gamma_{k,x}\epsilon_k\|x^k - y^k\|^2 + (\gamma_{k,x}^2(L_x^2 + (\epsilon_k)^2) + \gamma_{k,x}^{2-\tau} L_\theta^2)\|x^k - y^k\|^2 + \beta_k\gamma_{k,\theta}^2\nu_\theta^2 + \gamma_{k,x}^2\nu_x^2. \end{aligned}$$

By Lemma 3.2 and (A2-3),  $\sum_{k=0}^\infty \|y^k - y^{k-1}\| < \infty$ . and  $\sum_{k=0}^\infty \|y^k - y^{k-1}\|^2 < \infty$ . Therefore, by boundedness of  $X$ , (A2-3) and Lemma 2.3, we have that there exists a random variable  $V$  such that

$$\|x^k - y^{k-1}\|^2 + \beta_k\|\theta^k - \theta^*\|^2 \rightarrow V \quad a.s. \quad \text{as } k \rightarrow \infty.$$

and  $\sum_{k=0}^\infty 2\gamma_{k,x}\epsilon_k\|x^k - y^k\|^2 < \infty$ . Since  $\sum_{k=0}^\infty \gamma_{k,x}\epsilon_k = \infty$ , we get  $\|x^k - y^k\| \rightarrow 0$  a.s. as  $k \rightarrow \infty$ . This implies  $\|x^k - x^*\| \rightarrow 0$  a.s. as  $k \rightarrow \infty$ .  $\square$

**3.2. Diminishing and constant steplength error analysis.** In this section, we estimate the convergence rate of the proposed schemes. Analogous to Section 2.3, we obtain the optimal  $\mathcal{O}(1/K)$  rate estimate for the upper bound on the expected error in the solution  $x_K$  when  $F(\bullet; \theta^*)$  is strongly monotone in  $(\bullet)$ . In addition, when  $F(\bullet; \theta^*)$  is merely monotone and the variational inequality problem possesses the *minimum principle sufficiency* (MPS) property (See Lemma 3.3 for a definition of the MPS property), a rate estimate is still available by using averaging. If we replace  $\nabla_x f$  and  $\nabla_\theta g$  by  $F$  and  $G$ , respectively, in Theorem 2, then we obtain the following:

**THEOREM 5 (Rate estimate for strongly monotone  $F$ ).** Suppose (A1-3) and (A3) hold. Suppose  $\gamma_{x,k} = \lambda_x/k$  and  $\gamma_{\theta,k} = \lambda_\theta/k$  with  $\lambda_x > 1/\mu_x$  and  $\lambda_\theta > 1/(2\mu_\theta)$ . Let  $\mathbb{E}[\|F(x^k; \theta^k) + w^k\|^2] \leq M^2$  and  $\mathbb{E}[\|G(\theta^k) + v^k\|^2] \leq M_\theta^2$  for all  $x^k \in X$  and  $\theta^k \in \Theta$ . Suppose  $x^*$  is the unique solution to  $VI(X, \mathbb{E}[F(\bullet; \theta^*, \xi)])$ . Let  $\{x^k, \theta^k\}$  be computed via Algorithm 2. Then, the following hold:

$$\mathbb{E}[\|\theta^k - \theta^*\|^2] \leq \frac{Q_\theta(\lambda_\theta)}{k} \text{ and } \mathbb{E}[\|x^k - x^*\|^2] \leq \frac{Q_x(\lambda_x)}{k},$$

where  $Q_\theta(\lambda_\theta) \triangleq \max \{ \lambda_\theta^2 M_\theta^2 (2\mu_\theta \lambda_\theta - 1)^{-1}, \mathbb{E}[\|\theta^1 - \theta^*\|^2] \}$ ,  $Q_x(\lambda_x) \triangleq \max \{ \lambda_x^2 \widetilde{M}^2 (\mu_x \lambda_x - 1)^{-1}, \mathbb{E}[\|x^1 - x^*\|^2] \}$ ,

$$\text{and } \widetilde{M} \triangleq \sqrt{M^2 + \frac{L_\theta^2 Q_\theta(\lambda_\theta)}{\mu_x \lambda_x}}.$$

Next, we weaken the strong monotonicity of  $F$ , but assume that  $(\mathcal{P}_x^v(\theta^*))$  satisfies the MPS property, introduced in the following Lemma. Note that this property guarantees weak sharpness of the solution set; this is analogous to weak-sharpness of minima in optimization problems [12].

**LEMMA 3.3** (Theorem 4.3 in [38]). Let  $H : X \rightarrow \mathbb{R}^n$  be a mapping that is monotone over the compact polyhedral set  $X$ . Let  $X^*$  be the solution set of  $VI(X, H)$ . If the  $VI(X, H)$  possesses the minimum principle sufficiency (MPS) property, then there exists a positive number  $\alpha$  such that  $(x - x^*)^T H(x^*) \geq \alpha \text{dist}(x, X^*)$ ,  $\forall x \in X$ ,  $\forall x^* \in X^*$ , where  $\text{dist}(x, X^*) \triangleq \min_{x^* \in X^*} \|x - x^*\|$ . We say that the  $VI(X, H)$  possesses the MPS property if  $\Gamma(x^*) = X^*$  for every  $x^*$  in  $X^*$ , where  $\Gamma(x) = \arg \max_{y \in X} (x - y)^T H(x)$ .

By leveraging this property, we may estimate the convergence rate by using averaging as in Theorem 2.

**THEOREM 6 (Rate estimates under monotonicity of  $F$ ).** Suppose (A1-4) and (A3) hold. Suppose  $\mathbb{E}[\|x^k - x^*\|^2] \leq M_x^2$ ,  $\mathbb{E}[\|F(x^k; \theta^k) + w^k\|^2] \leq M^2$  and  $\mathbb{E}[\|G(\theta^k) + v^k\|^2] \leq M_\theta^2$  for all  $x^k \in X$  and  $\theta^k \in \Theta$ . Suppose  $X$  is a compact polyhedral set, the solution set  $X^*$  of  $VI(X, \mathbb{E}[F(\bullet; \theta^*, \xi)])$  is nonempty, and  $x^*$  is a point in  $X^*$ . Suppose  $VI(X, \mathbb{E}[F(\bullet; \theta^*, \xi)])$  possesses the MPS property. Let  $\{x^k, \theta^k\}$  be computed via Algorithm 2. For  $1 \leq i, t \leq k$ , we define  $v_t \triangleq \frac{\gamma_{x,t}}{\sum_{s=i}^k \gamma_{x,s}}$ ,  $\tilde{x}_{i,k} \triangleq \sum_{t=i}^k v_t x^t$  and  $D_X \triangleq \max_{x \in X} \|x - x^1\|$ . Suppose for  $1 \leq t \leq k$

$$\gamma_x = \sqrt{\frac{4D_X^2 + L_\theta^2 Q_\theta(\lambda_\theta)(1 + \ln k)}{(M^2 + M_x^2)k}},$$

where  $Q_\theta(\lambda_\theta) \triangleq \max \{ \lambda_\theta^2 M_\theta^2 (2\mu_\theta \lambda_\theta - 1)^{-1}, \mathbb{E}[\|\theta^1 - \theta^*\|^2] \}$ , and  $\gamma_{\theta,k} = \lambda_\theta/k$  with  $\lambda_\theta > 1/(2\mu_\theta)$ . Then there exists a positive number  $\alpha$  such that for  $1 \leq i \leq k$ :

$$\mathbb{E}[\alpha \text{dist}(\tilde{x}_{i,k}, X^*)] \leq C_{i,k} \sqrt{\frac{B_k}{k}},$$

where  $C_{i,k} = \frac{k}{k-i+1}$  and  $B_k = (4D_X^2 + L_\theta^2 Q_\theta(\lambda_\theta)(1 + \ln k))(M^2 + M_x^2)$ .

*Proof.* By using the same notation in Theorem 2 except that we replace  $\nabla_x f$  and  $\nabla_\theta g$  by  $F$  and  $G$ , respectively, we have from (2.20) that

$$(3.9) \quad a_{k+1} \leq a_k + \frac{1}{2}\gamma_{x,k}^2 M^2 - \gamma_{x,k} \mathbb{E}[(x^k - x^*)^T F(x^k; \theta^*)] - \gamma_{x,k} \mathbb{E}[(x^k - x^*)^T (F(x^k; \theta^k) - F(x^k; \theta^*))].$$

By Lemma 3.3, we have that there exists a positive number  $\alpha$  such that

$$(3.10) \quad \begin{aligned} \alpha \operatorname{dist}(x^k, X^*) &\leq (x^k - x^*)^T F(x^k; \theta^*) = (x^k - x^*)^T F(x^k; \theta^*) - (x^k - x^*)^T (F(x^k; \theta^*) - F(x^k; \theta^*)) \\ &\leq (x^k - x^*)^T F(x^k; \theta^*), \end{aligned}$$

where the last inequality follows from the monotonicity of  $F(\bullet; \theta^*)$  in  $(\bullet)$ . Combining (3.9) and (3.10),

$$(3.11) \quad \begin{aligned} \alpha \gamma_{x,k} \mathbb{E}[\operatorname{dist}(x^k, X^*)] &\leq \gamma_{x,k} \mathbb{E}[(x^k - x^*)^T F(x^k; \theta^*)] \\ &\leq a_k - a_{k+1} + \frac{1}{2}\gamma_{x,k}^2 M^2 - \gamma_{x,k} \mathbb{E}[(x^k - x^*)^T (F(x^k; \theta^k) - F(x^k; \theta^*))]. \end{aligned}$$

Next, we follow the same proof method in Theorem 2. We define  $v_t \triangleq \frac{\gamma_{x,t}}{\sum_{s=i}^k \gamma_{x,s}}$  and  $D_X \triangleq \max_{x \in X} \|x - x^1\|$ . It follows from (2.24) and (3.11) that

$$(3.12) \quad \mathbb{E} \left[ \alpha \sum_{t=i}^k v_t \operatorname{dist}(x^t, X^*) \right] \leq \frac{a_i + \frac{1}{2} \sum_{t=i}^k \gamma_{x,t}^2 (M^2 + M_x^2) + \frac{1}{2} L_\theta^2 Q_\theta(\lambda_\theta) (1 + \ln k)}{\sum_{t=i}^k \gamma_{x,t}}.$$

Next, we consider points given by  $\tilde{x}_{i,k} \triangleq \sum_{t=i}^k v_t x^t$ . Since  $F(x; \theta^*)$  is monotone in  $x$ , we have that  $X^*$  is convex, which implies that  $\operatorname{dist}(x, X^*)$  is convex in  $x$ . So, we get  $\operatorname{dist}(\tilde{x}_{i,k}, X^*) \leq \sum_{t=i}^k v_t \operatorname{dist}(x^t, X^*)$ . It follows from (2.25) and (3.12) that for  $1 \leq i \leq k$

$$(3.13) \quad \mathbb{E} [\alpha \operatorname{dist}(\tilde{x}_{i,k}, X^*)] \leq \frac{4D_X^2 + \sum_{t=i}^k \gamma_{x,t}^2 (M^2 + M_x^2) + L_\theta^2 Q_\theta(\lambda_\theta) (1 + \ln k)}{2 \sum_{t=i}^k \gamma_{x,t}}.$$

Suppose  $\gamma_{x,t} = \gamma_x$  for  $t = 1, \dots, k$ . If we follow the same proof method in Theorem 2, then we can get from (2.27) and (3.13) that

$$\mathbb{E} [\alpha \operatorname{dist}(\tilde{x}_{i,k}, X^*)] \leq C_{i,k} \sqrt{\frac{B_k}{k}}.$$

□

The following corollary is a special case of Theorem 6, an avenue that has been adopted in [40].

**COROLLARY 2 (Rate estimates under monotonicity of  $F$ ).** Suppose (A1-4) and (A3) hold. Suppose  $\mathbb{E}[\|x^k - x^*\|^2] \leq M_x^2$ ,  $\mathbb{E}[\|F(x^k; \theta^k) + w^k\|^2] \leq M^2$  and  $\mathbb{E}[\|G(\theta^k) + v^k\|^2] \leq M_\theta^2$  for all  $x^k \in X$  and  $\theta^k \in \Theta$ . Suppose  $X$  is a compact polyhedral set, the solution set  $X^*$  of  $VI(X, \mathbb{E}[F(\bullet; \theta^*, \xi)])$  is nonempty, and  $x^*$  is a point in  $X^*$ . Suppose  $VI(X, \mathbb{E}[F(\bullet; \theta^*, \xi)])$  possesses the MPS property. Let  $\{x^k, \theta^k\}$  be computed via Algorithm 2. For  $k/2 \leq t \leq k$ , we define  $v_t \triangleq \frac{\gamma_{x,t}}{\sum_{s=k/2}^k \gamma_{x,s}}$ ,  $\tilde{x}_{k/2,k} \triangleq \sum_{t=k/2}^k v_t x^t$  and  $D_X \triangleq \max_{x \in X} \|x - x^1\|$ . Suppose for  $1 \leq t \leq k$

$$\gamma_x = \sqrt{\frac{4D_X^2 + L_\theta^2 Q_\theta(\lambda_\theta)(1 + \ln 2)}{(M^2 + M_x^2)k}},$$

where  $Q_\theta(\lambda_\theta) \triangleq \max \{ \lambda_\theta^2 M_\theta^2 (2\mu_\theta \lambda_\theta - 1)^{-1}, \mathbb{E}[\|\theta^1 - \theta^*\|^2] \}$ , and  $\gamma_{\theta,k} = \lambda_\theta/k$  with  $\lambda_\theta > 1/(2\mu_\theta)$ . Then there exists a positive number  $\alpha$  such that

$$\mathbb{E}[\alpha \text{dist}(\tilde{x}_{k/2,k}, X^*)] \leq 2\sqrt{\frac{B}{k}},$$

where  $B = (4D_X^2 + L_\theta^2 Q_\theta(\lambda_\theta)(1 + \ln 2))(M^2 + M_x^2)$ .

*Proof.* When  $i = k/2$  where  $k$  is a positive even number, then by utilizing the same approach as in Corollary 1, inequality (3.13) becomes the following:

$$(3.14) \quad \mathbb{E}[\alpha \text{dist}(\tilde{x}_{k/2,k}, X^*)] \leq \frac{4D_X^2 + \sum_{t=k/2}^k \gamma_{x,t}^2 (M^2 + M_x^2) + L_\theta^2 Q_\theta(\lambda_\theta)(1 + \ln 2)}{2 \sum_{t=k/2}^k \gamma_{x,t}}.$$

Suppose  $\gamma_{x,t} = \gamma_x$  for  $t = 1, \dots, k$ . By utilizing the same techniques as in Theorem 6, then we obtain the following bound:

$$\mathbb{E}[\alpha \text{dist}(\tilde{x}_{k/2,k}, X^*)] \leq 2\sqrt{\frac{B}{k}},$$

where  $B \triangleq (4D_X^2 + L_\theta^2 Q_\theta(\lambda_\theta)(1 + \ln 2))(M^2 + M_x^2)$ .  $\square$

Next, we present a constant steplength error bound.

**PROPOSITION 5 (Constant steplength error bound).** Suppose (A3) holds. Suppose  $\gamma_{\theta,k} := \gamma_\theta$  and  $\gamma_{x,k} := \gamma_x$ . Suppose  $\mathbb{E}[\|x^k - x^*\|^2] \leq M_x^2$  and  $\mathbb{E}[F(x^k; \theta^k) + w^k]^2] \leq M^2$  for all  $x^k \in X$ . Suppose  $A_k \triangleq \frac{1}{2}\|x^k - x^*\|^2$  and  $a_k \triangleq \mathbb{E}[A_k]$ . Suppose  $X$  is a compact polyhedral set, the solution set  $X^*$  of  $VI(X, \mathbb{E}[F(\bullet; \theta^*, \xi)])$  is nonempty, and  $x^*$  is a point in  $X^*$ . Suppose  $VI(X, \mathbb{E}[F(\bullet; \theta^*, \xi)])$  possesses the MPS property. Let  $\{x^k, \theta^k\}$  be computed via Algorithm 1.

(i) Suppose (A1-3) holds. Then, the following holds:

$$\limsup_{k \rightarrow \infty} a_k \leq \frac{1}{2\mu_x} \gamma_x M^2 + \frac{1}{2} \frac{1}{\mu_x^2} L_\theta^2 \frac{\gamma_\theta \nu_\theta^2}{2\mu_\theta - \gamma_\theta C_\theta^2};$$

(ii) Suppose (A1-4) holds. Then, there exists a positive number  $\alpha$  such that:

$$\limsup_{k \rightarrow \infty} \mathbb{E}[\text{dist}(x^k, X^*)] \leq \frac{1}{\alpha} \left[ \frac{1}{2} \gamma_x M^2 + \frac{1}{2} \gamma_x^{1-\tau} M_x^2 + \frac{1}{2} \gamma_x^{\tau-1} L_\theta^2 \frac{\gamma_\theta \nu_\theta^2}{2\mu_\theta - \gamma_\theta C_\theta^2} \right],$$

where  $0 < \tau < 1$ .

*Proof.* If we replace  $\nabla_x f$  and  $\nabla_{\theta} g$  by  $F$  and  $G$  in Proposition 3, we obtain that

$$\limsup_{k \rightarrow \infty} \mathbb{E}[\|\theta^k - \theta^*\|^2] \leq \frac{\gamma_{\theta} \nu_{\theta}^2}{2\mu_{\theta} - \gamma_{\theta} C_{\theta}^2},$$

and the following can be derived based on the properties of  $F$ :

(i)  $F$  is strongly monotone:

$$\limsup_{k \rightarrow \infty} a_k \leq \frac{1}{2\mu_x} \gamma_x M^2 + \frac{1}{2} \frac{1}{\mu_x^2} L_{\theta}^2 \frac{\gamma_{\theta} \nu_{\theta}^2}{2\mu_{\theta} - \gamma_{\theta} C_{\theta}^2};$$

(ii)  $f$  is convex: From (3.11), for  $\gamma_{x,k} := \gamma_x$ , we have that there exists a positive number  $\alpha$  such that:

$$\begin{aligned} \alpha \gamma_x \mathbb{E}[\text{dist}(x^k, X^*)] &\leq \gamma_x \mathbb{E}[(x^k - x^*)^T F(x^k; \theta^*)] \\ &\leq a_k - a_{k+1} + \frac{1}{2} \gamma_x^2 M^2 - \gamma_x \mathbb{E}[(x^k - x^*)^T (F(x^k; \theta^k) - F(x^k; \theta^*))] \\ &\leq a_k - a_{k+1} + \frac{1}{2} \gamma_x^2 M^2 + \frac{1}{2} \gamma_x^{2-\tau} \mathbb{E}[\|x^k - x^*\|^2] + \frac{1}{2} \gamma_x^{\tau} \mathbb{E}[\|F(x^k; \theta^k) - F(x^k; \theta^*)\|^2] \\ &\leq a_k - a_{k+1} + \frac{1}{2} \gamma_x^2 M^2 + \frac{1}{2} \gamma_x^{2-\tau} M_x^2 + \frac{1}{2} \gamma_x^{\tau} L_{\theta}^2 \mathbb{E}[\|\theta^k - \theta^*\|^2], \end{aligned}$$

where  $0 < \tau < 1$ . It follows that

$$\begin{aligned} \alpha \gamma_x \mathbb{E}[\text{dist}(x^k, X^*)] &\leq \limsup_{k \rightarrow \infty} a_k - \limsup_{k \rightarrow \infty} a_{k+1} + \frac{1}{2} \gamma_x^2 M^2 + \frac{1}{2} \gamma_x^{2-\tau} M_x^2 + \frac{1}{2} \gamma_x^{\tau} L_{\theta}^2 \limsup_{k \rightarrow \infty} \mathbb{E}[\|\theta^k - \theta^*\|^2] \\ &\leq \frac{1}{2} \gamma_x^2 M^2 + \frac{1}{2} \gamma_x^{2-\tau} M_x^2 + \frac{1}{2} \gamma_x^{\tau} L_{\theta}^2 \frac{\gamma_{\theta} \nu_{\theta}^2}{2\mu_{\theta} - \gamma_{\theta} C_{\theta}^2}. \end{aligned}$$

It follows that

$$\limsup_{k \rightarrow \infty} \mathbb{E}[\text{dist}(x^k, X^*)] \leq \frac{1}{\alpha} \left[ \frac{1}{2} \gamma_x M^2 + \frac{1}{2} \gamma_x^{1-\tau} M_x^2 + \frac{1}{2} \gamma_x^{\tau-1} L_{\theta}^2 \frac{\gamma_{\theta} \nu_{\theta}^2}{2\mu_{\theta} - \gamma_{\theta} C_{\theta}^2} \right].$$

□

**4. Numerical results.** In this section, we apply the developed algorithms on a class of misspecified economic dispatch problems described in Section 4.1. In Section 4.2, we apply the proposed schemes for purposes of learning optimal solutions and the misspecified parameters. Note that the simulations were carried out on Tomlab 7.4. The complementarity solver **PATH** [18] was utilized for obtaining solutions to these problems which subsequently formed the basis for comparison.

**4.1. Problem description.** We consider a setting where there are  $N$  firms competing over a  $W$ -node network. Firm  $f$  may produce and sell its good at node  $i$ , where  $f = 1, \dots, N$  and  $i = 1, \dots, W$ . We assume that for a given firm  $f$ , the cost of generating  $x_{fi}$  units of power at node  $i$  is random and is given by  $c_{fi}(x_{fi}) = d_{fi}x_{fi}^2 + h_{fi}x_{fi} + \xi_{fi}$ , where  $d_{fi}$  and  $h_{fi}$  are positive parameters, and  $\xi_{fi}$  is a random variable with mean zero for all  $f$  and  $i$ . Furthermore, the generation level associated with firm  $f$  is bounded by its production capacity, which is denoted by  $\text{cap}_{fi}$ . The aggregate sales of all firms at node  $i$  has to satisfy the demand  $D_i$  at node  $i$ . A given firm can produce at any node and then sell at different nodes, provided that the aggregate production at all nodes matches the aggregate sales at all nodes for each firm. For simplicity,

we assume that there is no limit of sales at any node. Then, the resulting problem faced by the grid operator can be stated as follows:

$$(4.1) \quad \begin{aligned} & \min_{x_{fi} \geq 0} \quad \mathbb{E} \left[ \sum_{f=1}^N \sum_{i=1}^W c_{fi}(x_{fi}) \right] \\ & \text{subject to} \quad x_{fi} \leq \text{cap}_{fi}, \quad \text{for all } f, i \\ & \quad \sum_{f=1}^N x_{fi} = D_i. \end{aligned}$$

The resulting optimal solution is given by  $x^*$ . Suppose firm  $f$  generates  $y_{fi}$  units of power at node  $i$ . We use  $c_{fi}(y_{fi}) = d_{fi}(y_{fi})^2 + h_{fi}y_{fi} + \xi_{fi}$  to denote the cost associated with firm  $f$  at node  $i$ . The operator will solve the following (regularized) problem to estimate  $c_{fi}$  and  $d_{fi}$ :

$$(4.2) \quad \min_{\{d_{fi}, h_{fi}\} \in \Theta} \mathbb{E} [(d_{fi}(y_{fi})^2 + h_{fi}y_{fi} - c_{fi}(y_{fi}))^2 + \mu_\theta d_{fi}^2 + \mu_\theta h_{fi}^2].$$

The resulting optimal solution is given by  $\theta^*$ . We assume that  $y_{fi}$  is distributed as per a uniform distribution and is specified by  $y_{fi} \sim U[0, \text{cap}_{fi}]$ , while that the noise  $\xi_{fi}$  is distributed as per a uniform distribution and is specified by  $\xi_{fi} \sim U[-\theta_{fi}^*/2, \theta_{fi}^*/2]$ .

**4.2. Results.** In this subsection, we employ Algorithm 1 proposed in Section 2 for learning parameters and computing optimal solutions. We will examine the behavior and error bounds of the algorithm.

**4.2.1. Behavior of the algorithm.** In this part, we consider a special case when  $N = 5$  and  $W = 5$ . Suppose, the noise  $\xi$  is distributed as per a uniform distribution and is specified by  $\xi \sim U[-\theta^*/2, \theta^*/2]$ . Suppose the steplength sequences  $\{\gamma_{k,x}\}$  and  $\{\gamma_{k,\theta}\}$  are chosen according to Proportion 2:  $\gamma_{k,x} = 1/k$  and  $\gamma_{k,\theta} = 40/k$ . Figure 4.1(a) illustrates the scaled error of the learning scheme when the number of steps increases.

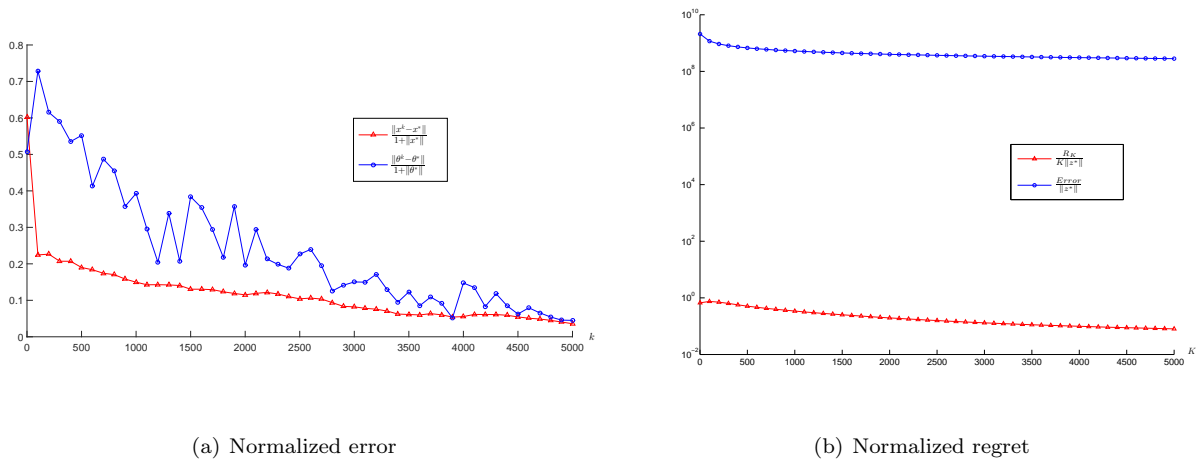


FIG. 4.1. Computing  $x^*$  and learning  $\theta^*$  ( $\xi \sim U[-\theta^*/2, \theta^*/2]$ ,  $N = 5$ ,  $W = 5$ )

**4.2.2. Error bounds.** In this part, we examine the errors of the algorithm and compare them with the theoretical error bounds proposed in Section 2. Suppose, the noise  $\xi$  is distributed as per a uniform distribution and is specified by  $\xi \sim U[-\theta^*/2, \theta^*/2]$ .

- (a) In the strongly convex regime, suppose the steplength sequences  $\{\gamma_{k,x}\}$  and  $\{\gamma_{k,\theta}\}$  are chosen according to Proportion 2:  $\gamma_{k,x} = 1/k$  and  $\gamma_{k,\theta} = 40/k$ . We use ERR to denote the theoretical error provided in Proportion 2. The algorithm was terminated at  $K = 10000$ . Table 4.1 (L) shows the scaled errors of the learning scheme.
- (b) In the merely convex regime, suppose the steplength  $\gamma_x$  and the steplength sequence  $\{\gamma_{k,\theta}\}$  are chosen according to Theorem 2:  $\gamma_x$  is chosen by Table 4.1 (R) and  $\gamma_{k,\theta} = 40/k$ . We use ERR to denote the theoretical error provided in Theorem 2 while  $z^*$  denotes  $f(x^*; \theta^*)$ . The algorithm was terminated at  $K = 10000$  and Table 4.1(R) shows the scaled errors of the learning scheme.
- (c) Suppose the steplength sequences  $\{\gamma_{k,x}\}$  and  $\{\gamma_{k,\theta}\}$  are chosen according to Theorem 3:  $\gamma_{k,x} = k^{-\alpha}$  and  $\gamma_{k,\theta} = 40/k$ . We employ ERR to denote the theoretical error provided in Theorem 3 while  $z^*$  denotes  $f(x^*; \theta^*)$ . The algorithm was terminated after  $K = 10000$  iterations. Figure 4.1(b) illustrates the scaled regret and scaled theoretical error of the learning scheme when the number of steps increases ( $\alpha = \beta = 0.5$ ). Table 4.2 shows the scaled theoretical error of the learning scheme for different chosen  $\gamma_{k,x} = k^{-\alpha}$  with  $\alpha = 0.5, 0.6, 0.7, 0.8, 0.9$  when  $\beta = 0.5$ . We see that when  $\alpha$  changes, error bounds change marginally primarily because the last term in Theorem 3 dominates the bound.

TABLE 4.1

*Learning  $x^*$  and  $\theta^*$  in a strongly convex (L) and convex (R) regime:  $\xi \sim U[-\theta^*/2, \theta^*/2]$*

N	W	$\frac{\mathbb{E}[\ x^K - x^*\ ]}{1 + \ x^*\ }$	$\frac{\text{ERR}}{1 + \ x^*\ }$	$\frac{\mathbb{E}[\ \theta^K - \theta^*\ ]}{1 + \ \theta^*\ }$	$\frac{\text{ERR}}{1 + \ \theta^*\ }$	N	W	$\frac{\mathbb{E}[f(\bar{x}_{1,K}; \theta^K) - z^*]}{1 + \ z^*\ }$	$\frac{\text{ERR}}{1 + \ x^*\ }$	$\gamma_x$
10	2	$7.3 \times 10^{-3}$	$9.2 \times 10^9$	$4.8 \times 10^{-2}$	$3.7 \times 10^4$	10	2	$1.9 \times 10^{-1}$	$2.5 \times 10^9$	72
10	4	$3.7 \times 10^{-2}$	$2.1 \times 10^{10}$	$4.9 \times 10^{-2}$	$3.1 \times 10^4$	10	4	$6.5 \times 10^{-2}$	$1.1 \times 10^5$	93
10	6	$3.8 \times 10^{-2}$	$7.8 \times 10^{10}$	$4.7 \times 10^{-2}$	$8.3 \times 10^4$	10	6	$2.7 \times 10^{-1}$	$2.6 \times 10^5$	127
10	8	$1.7 \times 10^{-2}$	$9.1 \times 10^{10}$	$4.8 \times 10^{-2}$	$8.5 \times 10^4$	10	8	$1.3 \times 10^{-1}$	$1.7 \times 10^5$	131
10	10	$2.4 \times 10^{-2}$	$1.2 \times 10^{11}$	$4.3 \times 10^{-2}$	$8.6 \times 10^4$	10	10	$1.4 \times 10^{-1}$	$2.6 \times 10^5$	133

TABLE 4.2

*Investigation of regret when learning  $x^*$  and  $\theta^*$  in a stochastic convex regime:  $\xi \sim U[-\theta^*/2, \theta^*/2]$ ,  $N = 5$ ,  $W = 5$*

$\alpha$	$\frac{R_K}{K \ z^*\ }$	$\frac{\text{ERR}}{\ z^*\ }$
0.5	$4.8 \times 10^{-2}$	$3.1 \times 10^8$
0.6	$3.3 \times 10^{-2}$	$3.1 \times 10^8$
0.7	$2.3 \times 10^{-2}$	$3.1 \times 10^8$
0.8	$1.8 \times 10^{-2}$	$3.1 \times 10^8$
0.9	$1.5 \times 10^{-2}$	$3.1 \times 10^3$

**5. Concluding remarks.** Traditionally, much of the field of optimization has been defined by problems in which the functions and sets are known to the decision-maker. However, as problems grow in their reliance on data, such knowledge cannot be taken for granted. We consider one such instance of such problems where functions may be misspecified and the associated vector may be learnt through the parallel solution of a suitably defined problem. It is worth emphasizing the problem in the *full* space of learning and optimization variables is a challenging (non-monotone) stochastic variational problem for which no first-order methods are currently available. Yet, by leveraging the structure of the problem, we show that such problems can indeed be efficiently solved.

We consider a problem of solving a stochastic optimization problem in which the objective is parameterized by a vector that can be learnt by solving a suitably defined learning problem, captured by a stochastic optimization problem. In both strongly convex and merely convex regimes, we develop a set of coupled stochastic approximation schemes which produces a sequence of iterates that are shown to converge to the solution and unknown parameter in an almost sure sense. Additionally, we provide rate estimates for the prescribed schemes in both strongly convex and convex regimes. Through an analysis of the rate of convergence under a diminishing steplength setting, it is seen that the optimal rate of convergence is observed in strongly convex problems while in convex regimes, we see a degradation introduced by learning from  $\mathcal{O}\left(\frac{1}{\sqrt{K}}\right)$  to  $\mathcal{O}\left(\frac{\sqrt{\ln(K)}}{\sqrt{K}}\right)$ . This degradation is seen to disappear if the averaging window is modified appropriately. Similar rate statements are also provided in a constant steplength regime. In fact, we may also cast this problem as an online decision-making problem where a decision-maker sees a collection of misspecified functions. In a stochastic regime, we observe that an upper bound on the average regret can be shown to decay at a rate no worse than  $\mathcal{O}\left(\frac{\ln K}{\sqrt{K}}\right)$  for a suitably chosen steplength.

Unfortunately, the optimization-based model cannot accommodate settings where there is misspecification in the constraints or, more generally, if the associated decision-making problem is an equilibrium problem. Motivated by this gap, we consider a misspecified stochastic variational inequality problem and propose analogous stochastic approximation schemes for computation and learning. To resolve the challenge associated with merely monotone maps, we employ (Tikhonov) regularized counterparts for which almost-sure convergence statements can be provided. Additionally, we provide rate statements for constant and diminishing steplength regimes, of which the latter requires imposing a suitable weak-sharpness assumption on the original problem. Again, it is seen that while the schemes display the optimal rate of convergence under strongly monotone regimes, a degradation in the rate is seen in the monotone regime.

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#### Appendix A. Theorem 1 in [50].

DEFINITION 1. Given an algorithm  $A$ , a convex set  $F \subseteq \mathbb{R}^n$  and an infinite sequence  $\{c^1, c^2, \dots\}$  where each  $c^t : F \rightarrow \mathbb{R}$  is a convex function, if  $\{x^1, x^2, \dots\}$  are the vectors selected by  $A$ , then the cost of  $A$  until time  $T$  is defined as  $C_A(T) = \sum_{t=1}^T c^t(x^t)$ . The cost of a static feasible solution  $x \in F$  until time  $T$  is defined as  $C_x(T) = \sum_{t=1}^T c^t(x)$ . The regret of algorithm  $A$  until time  $T$  is defined as  $R_A(T) = C_A(T) - \min_{x \in F} C_x(T)$ .

The Greedy Projection algorithm proposed in [50] is as follows.

ALGORITHM 4 (Greedy Projection). Select an arbitrary  $x^1 \in F$  and a sequence of learning rates  $\eta_1, \eta_2, \dots \in \mathbb{R}^+$ . In time step  $t$ , after receiving a cost function, select the next vector  $x^{t+1}$  according to:

$$x^{t+1} = \Pi_F(x^t - \eta_t \nabla c^t(x^t)).$$

Then, we have the following result.

THEOREM 7 (Theorem 1 in [50]). If  $\eta_t = t^{-1/2}$ , the regret of the Greedy Projection algorithm is:

$$R_G(T) \leq \frac{\|F\|^2 \sqrt{T}}{2} + \left( \sqrt{T} - \frac{1}{2} \right) \|\nabla c\|^2,$$

where  $\|F\| \triangleq \max_{x, y \in F} d(x, y)$  and  $\|\nabla c\| \triangleq \max_{x \in F, t \in \{1, 2, \dots\}} \|\nabla c^t(x)\|$ .

*Proof sketch:* The regret of the Greedy Projection algorithm can be bounded as follows:

$$R_G(T) \leq \frac{\|F\|^2}{2\eta_T} + \frac{\|\nabla c\|^2}{2} \sum_{t=1}^T \eta_t.$$

The result can be immediately obtained when  $\eta_t = t^{-1/2}$ . ■